Galois and θ Cohomology

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Notes: www.liegroups.org/talks

Galois Cohomology

F local $\quad \Gamma = \mathrm{Gal}(\overline{F}/F)$ $G = G(\overline{F})$ defined over F

$G(F) = G(\overline{F})^{\Gamma}$

$H^i(\Gamma, G) =$ Galois cohomology $\quad$ (group cohomology)

$i = 0, 1$ if G is not abelian

Example: $G(F) = GL(n, F): \quad H^1(\Gamma, G) = 1$

$(GL(1, F) = F^*$: Hilbert's Theorem 90)

Example: $G = SO(V)$:

$H^1(\Gamma, G) = \{$ non-degenerate quadratic forms of same dimension and discriminant as $V\}$

Example: $G = Sp(2n, F)$

$H^1(\Gamma, G) = \{$ non-degenerate symplectic forms, dim. $2n\} = 1$

Rational Forms

Basic Fact:

$$\{\text{rational (inner) forms of } G\} \longleftrightarrow H^1(\Gamma, G_{\text{ad}})$$

(Inner: $\sigma'\sigma^{-1}$ is inner)

NB: (for the experts): equality of rational forms is by the action of G, not $\text{Aut}(G)$ (Borel: $\text{Image}(H^1(\Gamma, G_{\text{ad}}) \to H^1(\Gamma, \text{Aut}(G_{\text{ad}})))$

Theorem (Kneser): F p-adic, G simply connected $\Rightarrow H^1(\Gamma, G) = 1$

Not true over $\mathbb{R}$ ... $G(\mathbb{R}) = SU(2)$, $\quad H^1(\Gamma, G) = \mathbb{Z}/2\mathbb{Z})$

Problem: Calculate $H^1(\Gamma, G)$ G simply connected, defined over $\mathbb{R}$

This fact plays a role in statements about the trace formula, functoriality, packets. . .

Real case

$F = \mathbb{R}$, $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle \sigma \rangle$

$H^0(\Gamma, G) = G(\mathbb{R})$

$H^1(\Gamma, G) = \{g \in G \mid g\sigma(g) = 1\}/[g \to xg\sigma(x^{-1})]$

Write $H^i_\sigma(\Gamma, G)$

Digression: H =torus, $\widehat{H}^i(\Gamma, H)$ Tate cohomology

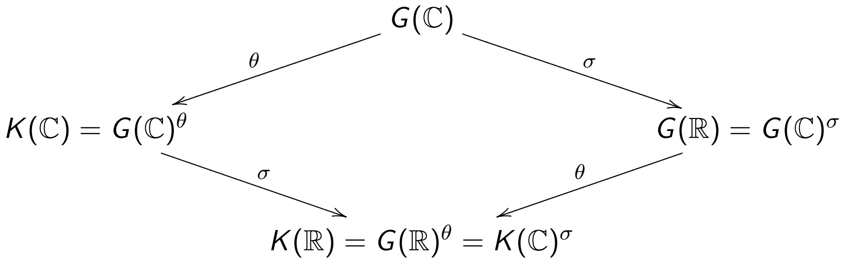
$\widehat{H}^0(\Gamma, H) = H(\mathbb{R})/H(\mathbb{R})^0$

Question: Is it possible (and a good idea?) to define $\widehat{H}^i(\Gamma, G)$ ($i = 0, 1$) in such a way that $\widehat{H}^0(\Gamma, G) = G(\mathbb{R})/G(\mathbb{R})^0$?

Real Forms: σ and θ pictures

Cartan: classify real forms by their Cartan involution: a real form is determined by its maximal compact subgroup $K(\mathbb{R})$ - in fact by $K(\mathbb{C}) = G(\mathbb{C})^\theta$

θ is a holomorphic involution.



Real Forms: σ and θ pictures

$$
\begin{array}{ccccc}
 & & GL(n,\mathbb{C}) & & \\
 & \overset{\theta}{\swarrow} & & \overset{\sigma}{\searrow} & \\
GL(p,\mathbb{C}) \times GL(q,\mathbb{C}) & & & & U(p,q) \\
 & \underset{\sigma}{\searrow} & & \underset{\theta}{\swarrow} & \\
 & & U(p) \times U(q) & &
\end{array}
$$

Theorem: (Cartan)

$$\{\sigma \mid \sigma \text{ antiholomorphic}\}/G \longleftrightarrow \{\theta \mid \sigma \text{ holomorphic}\}/G$$

$$\sigma \longrightarrow \theta = \sigma\sigma_c$$

$$\sigma = \theta\sigma_c \longleftarrow \theta$$

Real Forms: σ and θ pictures

σ, θ pictures are deeply embedded in representation theory

$\sigma : G(\mathbb{R})$ acting on a Hilbert space

$\theta : (\mathfrak{g}, K)$ modules $\mathfrak{g}, K$ both complex

Matsuki duality (later): $X = G(\mathbb{C})/B(\mathbb{C})$

$$G(\mathbb{R})\backslash X \longleftrightarrow K(\mathbb{C})\backslash X$$

Kostant-Sekiguchi correspondence (nilpotent $G(\mathbb{R}), K(\mathbb{C})$ orbits)

Real Forms: θ cohomology

θ holomorphic:

Definition: $H^i_\theta(\mathbb{Z}_2, G)$: group cohomology with $\mathbb{Z}_2$ acting by θ

$$H^0_\theta(\mathbb{Z}_2, G) = K$$

(remember $K = K(\mathbb{C})$)

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g\theta(g) = 1\}/[g \to xg\theta(x^{-1})]$$

Basic Point: $H^1_\theta(\mathbb{Z}_2, G)$ is much easier to compute than $H^1_\sigma(\Gamma, G)$

Example: $\theta = 1$:

$$H^1_\theta(\mathbb{Z}_2, G) = \{g \mid g^2 = 1\}/G = \{h \in H \mid h^2 = 1\}/W$$

$G(\mathbb{R})$ compact
(Serre): $H^1(\Gamma, G) = H^1(\Gamma, G(\mathbb{R})) = \{h \in H(\mathbb{R}) \mid h^2 = 1\}/W$

Real Forms: Example

$\theta = 1$, $H^1(\mathbb{Z}_2, G) = H_2/W$:

Exercise:

$G = E_8$, R=root lattice, $|R/2R| = 256$

$$|H^1_\theta(\mathbb{Z}_2, G)| = |(R/2R)/W| = 3 \quad (1 + 120 + 135 = 256)$$

Galois and θ cohomology

$H^1_\theta(\mathbb{Z}_2, G) \quad H^1_\sigma(\Gamma, G)$

Cartan's Theorem can be stated: $\sigma \leftrightarrow \theta \Rightarrow$

$$H^1_\sigma(\Gamma, G_{\mathrm{ad}}) \simeq H^1_\theta(\mathbb{Z}_2, G_{\mathrm{ad}})$$

Question: drop the adjoint condition?

Theorem: G connected reductive,

σ antiholomorphic, θ holomorphic
$\sigma \leftrightarrow \theta$ (in the sense of Cartan; i.e. defining the same real form)
There is a canonical isomorphism:

$$\boxed{H^1_\sigma(\Gamma, G) \simeq H^1_\theta(\mathbb{Z}_2, G)}$$

Sketch of proof

(1) H torus: $1 \to H_2 \to H \xrightarrow{z^2} H \to 1$

$|\Gamma = 2| \Rightarrow$

$H^1_\sigma(\Gamma, H) \simeq H^1_\sigma(\Gamma, H_2)$

$H^1_\theta(\mathbb{Z}_2, H) \simeq H^1_\theta(\mathbb{Z}_2, H_2)$

and $\theta|_{H_2} = \sigma|_{H_2}$

$$H^1_\sigma(\Gamma, H) \simeq H^1_\sigma(\Gamma, H_2) = H^1_\theta(\mathbb{Z}_2, H_2) \simeq H^1_\theta(\mathbb{Z}_2, H)$$

(2) H_f a fundamental (most compact) Cartan subgroup;

$$H^1_\sigma(\Gamma, H_f) \twoheadrightarrow H^1_\sigma(\Gamma, G)$$

(easy: every semisimple elliptic element is conjugate to an element of H_f)

Sketch of proof (continued)

(3) $W_i(H)$ = Weyl group of imaginary roots,

$$H^1_\sigma(\sigma, H)/W_i(H) \hookrightarrow H^1_\sigma(\Gamma, G)$$

This is non-trivial but standard:

it comes down to $(G/P)(F) = G(F)/P(F))$ (Borel-Tits) and there is only one conjugacy class of compact Cartan subgroups (very special to $\mathbb{R}$)

Equivalently: over $\mathbb{R}$ stable conjugacy of Cartan subgroups is equivalent to ordinary conjugacy (Shelstad) (false in the p-adic case)

$$\boxed{H^1_\sigma(\Gamma, G) \simeq H^1_\sigma(\Gamma, H_f)/W_i}$$

Theorem (Borovoi): $H^1_\sigma(\Gamma, G) \simeq H^1_\sigma(\Gamma, H_f)/W^\sigma$

Exactly same argument holds for θ-cohomology:

$$\boxed{H^1_\theta(\mathbb{Z}_2, G) \simeq H^1_\theta(\mathbb{Z}_2, H_f)/W_i}$$

Two versions of the rational Weyl group

$W_\sigma = \mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$

$W_\theta = \mathrm{Norm}_{K(\mathbb{C})}(H(\mathbb{C}))/H(\mathbb{C}) \cap K(\mathbb{C})$

Theorem (well known, see Warner): $W_\sigma \simeq W_\theta$

Applications

proof:

$$1 \to H \to N \to W \to 1$$

$$1 \to H^\sigma \to N^\sigma \to W^\sigma \to H^1_\sigma(\Gamma, H)$$

$$\begin{array}{ccccccc}
1 & \longrightarrow & W_\sigma = N^\sigma / H^\sigma & \longrightarrow & W^\sigma & \longrightarrow & H^1_\sigma(\Gamma, H) \\
 & & \downarrow & & \downarrow = & & \downarrow \simeq \\
1 & \longrightarrow & W_\theta = N^\theta / H^\theta & \longrightarrow & W^\theta & \longrightarrow & H^1_\theta(\Gamma, H)
\end{array}$$

Applications

Matsuki Correspondence of Cartan subgroups

Theorem (Matsuki): There is a canonical bijection

$$\{\sigma\text{-stable } H\}/G(\mathbb{R}) \leftrightarrow \{\theta\text{-stable } H\}/K$$

Proof in quasisplit case:

$$LHS = H^1_\sigma(\Gamma, W) \simeq H^1_\theta(\mathbb{Z}_2, W) = RHS$$

(general: $H^1_\sigma(\Gamma, N) \simeq H^1_\theta(\mathbb{Z}_2, N) \dots$)

Applications: Strong real forms

For simplicity: assume equal rank inner class

Definition (ABV) A strong real form of G is G-conjugacy class of $x \in G$ satisfying $x^2 \in Z(G)$.

$\{\text{strong real forms}\} \twoheadrightarrow \{\text{real forms}\}$ (bijection if G is adjoint)

$$x \to \theta_x = \text{int}(x) \text{ (conjugation by } x\text{)}$$

Pure Real forms: $x^2 = 1$

$$\begin{array}{ccc}
\{\text{strong real forms}\} & \twoheadrightarrow & H^1(\Gamma, G_{\text{ad}}) = \{\text{real forms}\} \\
\uparrow & & \uparrow \\
H^1(\Gamma, G) = \text{pure real forms} & \twoheadrightarrow & [\text{image}]
\end{array}$$

Problem:

1) Give a cohomological definition of strong real forms

2) Define "strong rational forms" of p-adic groups
(Kaletha): $H^1(u \to W, Z \to Z) =$strong real forms in real case

Applications: Strong real forms

Strong Real Forms:

$$x \to \text{inv}(x) = x^2 \in Z^\Gamma$$

Real forms:

$$\text{inv} : H^1(\Gamma, G_{\text{ad}}) \to H^2(\Gamma, Z) = \widehat{H}^0(\Gamma, Z) = Z^\Gamma/(1+\sigma)Z$$

Theorem: Given $\sigma \to \text{inv}(\sigma) \in Z^\Gamma/(1+\sigma)Z \to$ (choose) $z \in Z^\Gamma$

$$\boxed{H^1(\Gamma, G) \leftrightarrow \{\text{strong real forms of type } z\}}$$

$\rightarrow$: classical Galois cohomology interpretation of strong real forms

$\leftarrow$: compute $H^1(\Gamma, G)$(the right hand side is easy)

Applications: Strong real forms

Corollary:

$$\{\text{strong real forms}\} \longleftrightarrow \bigcup_{z \in S} H^1_{\sigma_z}(\Gamma, G)$$

$S = Z^\Gamma/(1+\sigma)Z$

$S \ni z \to \sigma_z \ (\sigma_z \leftrightarrow \theta_x \to x^2 = z)$

Application: Computing $H^1(\Gamma, G)$

Compute {strong real forms of type z}

(equal rank case):

$$H^1(\Gamma, G) \simeq \{g \in G \mid g^2 = z\}/G = \{h \in H \mid h^2 = z\}/W$$

(z depends on the real form)

Example:

$G = Sp(2n, \mathbb{R}) \quad x = \mathrm{diag}(i, \dots, i, -i, \dots, -i) \quad z = -I$:

$$H^1_\sigma(\Gamma, G) = \{g \mid g^2 = -I\}/G = \{\mathrm{diag}(\pm i, \dots, \pm i)\}/W = 1$$

Example:

$G = Spin(p, q)$

$SO(p, q)$: $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$ (classifying quadratic forms)

$Spin(p, q)$: $\lfloor \frac{p+q}{4} \rfloor + \delta(p, q) \quad \delta(p, q) = 0, 1, 2, 3$