Atlas of Lie Groups and Representations

www.liegroups.org

Computing Unipotent Representations

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Slides available at: www.liegroups.org

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Theorem: Suppose $G(\mathbb{R})$ is the real form of $G(\mathbb{C})$. Fix a regular infinitesimal character γ. Then there is a canonical bijection:

$\{$irreducible representations of $G(\mathbb{R})$ with infinitesimal character $\gamma\}$

and

$\{(H(\mathbb{R}), \Gamma) \mid H(\mathbb{R}) \text{ is a Cartan subgroup}, \Gamma \in \widehat{H(\mathbb{R})}, d\Gamma \sim_W \gamma\}/G(\mathbb{R})$

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Arthur conjectured that for each such Ψ there should be a finite set

$$\Pi(\Psi) \subset \Pi(G)_{adm}$$

satisfying various properties, including "stability" and unitarity.

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This is a (weak) Arthur packet of special unipotent representations¡

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10) $\Pi(\mathcal{O}^\vee)$ is the set of (non-zero) irreducible representations obtained this way.

The Algorithm: Primitive Ideals

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The columns of $P_{\mathcal{C}^\vee}$ correspond to the irreducible representations in the cell. Two such representations have the same primitive ideal $\Leftrightarrow$ the corresponding columns are multiples of each other.

[Interlude: some examples]

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What David Vogan talked about this morning was part of an algorithm to compute $\mathrm{AV}(\pi)$.