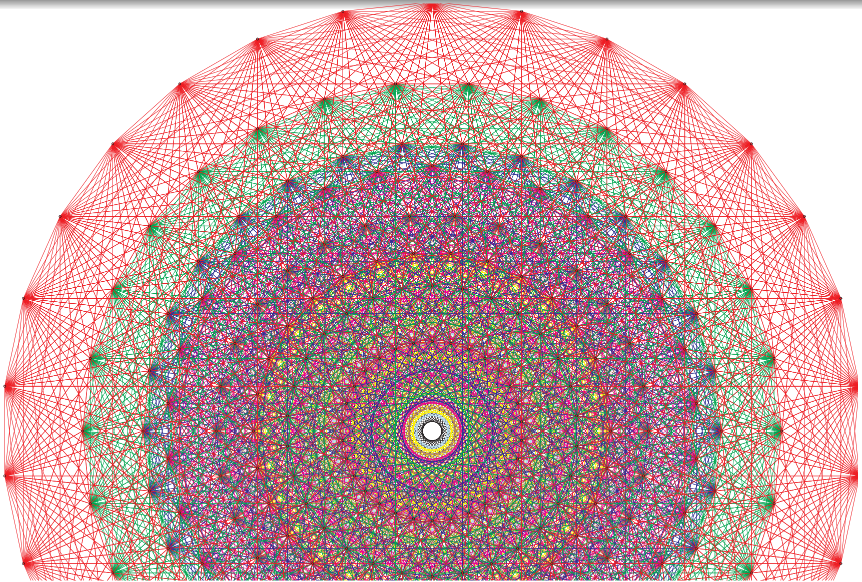
Atlas de Groups de Lie et Représentations

www.liegroups.org

E_8



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Funded by the National Science Foundation
American Institute of Mathematics

Atlas Workshop, July 2007
Palo Alto, California

E_8 is a Lie group

Lie groups are the mathematics of Symmetry

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Evariste Galois
France, 1811-1832
Groups

Sophus Lie
Norway, 1842-1899
Lie groups

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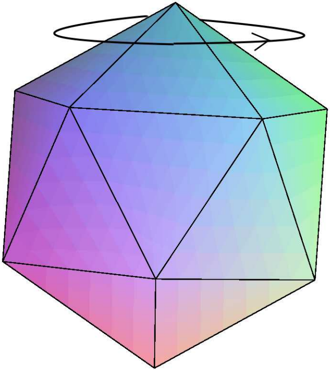
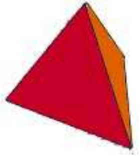
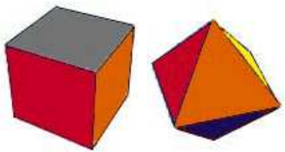
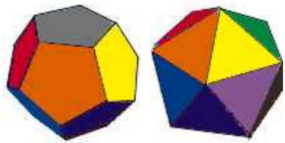


Symmetry Groups of the Platonic Solids 1800s

Platonic Solid	Symmetry group	Number of symmetries
	A_4 (even 4-permutations)	12
	S_4 (4-permutations)	24
	A_5 (even 5-permutations)	60

Applications of Symmetry

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- Mathematics (geometry, number theory, algebra. . .)
- Architecture, painting, textiles, music. . .

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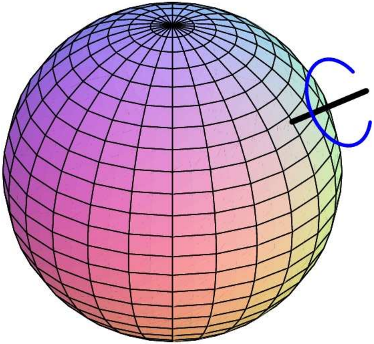

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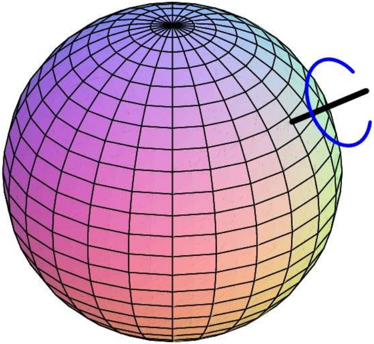

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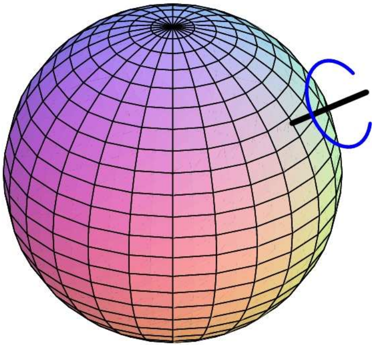

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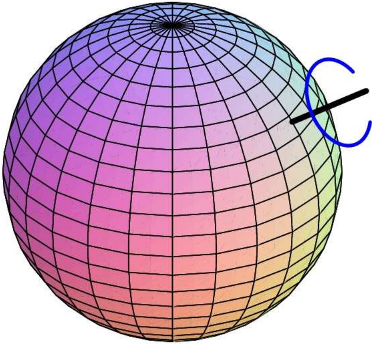

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This is the Rotation Group $SO(3)$, a 3 dimensional Lie group

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The periodic table is explained by representations of $SO(3)$

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5-dimensional cube

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$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The symmetry group of a cone:

Example: representations of the circle

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a representation of the Lie group $\{e^{i\theta} \,|\, 0 \leq \theta < 2\pi\}$ (the circle)

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The **Unitary Dual** of G is the collection of all of its irreducible unitary representations.

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This is abstract paper and pencil mathematics: computers have been of very little use.

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Goal of the Atlas of Lie Groups and Representations:

Use computers to help find the Unitary Dual

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- It requires new methods in computer science (unprecedented problems in algorithms and computation)

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Fokko du Cloux
Université de Lyon
(author of Coxeter software)

Abstract Mathematics
Lie Groups
Representation Theory

Abstract Mathematics	$\rightarrow$	Algorithm
Lie Groups		Combinatorial Set
Representation Theory		

What Fokko did 2003-2005

Abstract Mathematics	→	Algorithm	→	Software
Lie Groups		Combinatorial Set		C++ code
Representation Theory				

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The first arrow requires someone with very high level knowledge of both the mathematics and computers.

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Character table of A_5

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 3 & -1 & 0 & \tau & \overline{\tau} \\ 3 & -1 & 0 & \overline{\tau} & \tau \\ 4 & 0 & 1 & -1 & -1 \\ 5 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$\tau =$ Golden Ratio $\frac{1+\sqrt{5}}{2}$

$\overline{\tau} = \frac{1-\sqrt{5}}{2}$

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Just like for the genome, it can be very hard to extract this information: difficult problems in data mining

Here are some Lie groups

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(3) The symplectic group $Sp(2n)$ (arising in quantum mechanics):

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Group	Dimension
G_2	14
F_4	52
E_6	78
E_7	133
E_8	248

These are the **exceptional groups**

Some of the most complicated and fascinating objects in mathematics

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Some physicists think that E_8 plays an important role in mathematical physics and string theory: as a symmetry group of the laws of the universe

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The KLV matrix has 453,060 rows and columns

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In November 2005 Fokko computed the KLV matrix for all exceptional groups except E_8.

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In June 2006 Marc switched from other atlas tasks to working on E_8

May 2006

By May of 2006, Fokko was confined to his bed in Lyon. With help from friends and his dedicated life assistant Ange he continued to work on the software, using a video projector pointed at the ceiling, operated remotely by his collaborators.

Input: graph S with 453,060 vertices (one for each irreducible representation of E_8)

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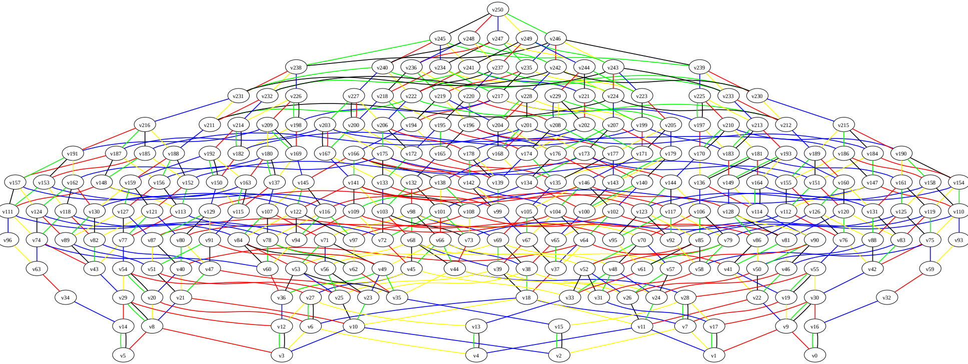
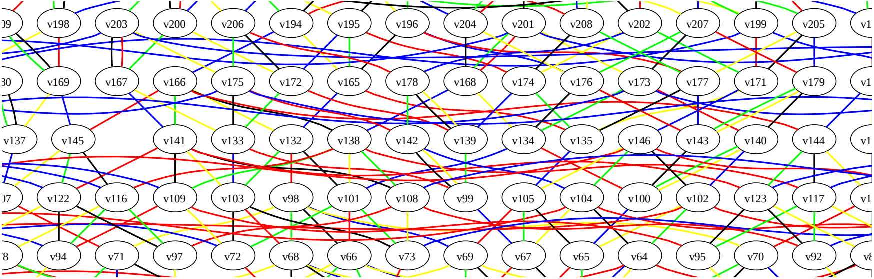


Graph for $SO(5,5)$ with 251 vertices



Closeup of $SO(5,5)$ graph

Output: Matrix $M = M(x, y)$ of KLV polynomials, with one row and column for every $x \in S$

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$$M(x, y) = 1 + q + 37q^7 + 19q^{22} + 101q^{31}$$

(degree ≤ 31)

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M(0,0)				
M(0,1)	←M(1,1)			
M(0,2)	←M(1,2)	←M(2,2)		
M(0,3)	←M(1,3)	←M(2,3)	←M(3,3)	
M(0,4)	←M(1,4)	←M(2,4)	←M(3,4)	←M(4,4)
...				

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Average number of non-zero terms: 150

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NOT parallelizable

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$453,060^2 = 205,263,363,600$ (205 billion)

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With some luck, and hard work, it looks like we'll need

1,000 gigabytes of RAM

(your PC has about 1 gigabyte of RAM)

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64 gigabytes of RAM/75 GB of swap/16 processors

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Should we buy such a machine, for about $150,000?

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Do calculation 4 times: mod 251, mod 253, mod 255, and mod 256

Combine the answer using the Chinese Remainder Theorem:

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$$\left.\begin{array}{ll} \text{mod} & 251 \\ \text{mod} & 253 \\ \text{mod} & 255 \\ \text{mod} & 256 \end{array}\right\} \rightarrow \text{mod} \quad 4,145,475,840$$

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Date	mod	Status	Result
Dec. 6	251	crash	
Dec. 19	251	complete	16 hours
Dec. 22	256	crash	
Dec. 22	256	complete	11 hours
Dec. 26	255	complete	12 hours

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Date	mod	Status	Result
Dec. 6	251	crash	
Dec. 19	251	complete	16 hours
Dec. 22	256	crash	
Dec. 22	256	complete	11 hours
Dec. 26	255	complete	12 hours
Dec. 27	253	crash	

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Dec. 6	251	crash	
Dec. 19	251	complete	16 hours
Dec. 22	256	crash	
Dec. 22	256	complete	11 hours
Dec. 26	255	complete	12 hours
Dec. 27	253	crash	
Jan. 3	253	complete	12 hours

We now have 132 gigabytes of data
(19 gigabytes data + 14 gigabytes index)$\times$4

```
-rw-r--r-- 1 root atlas  19G Jan  9  2007 E8coef-mod251
-rw-r--r-- 1 root atlas  19G Jan  8  2007 E8coef-mod253
-rw-r--r-- 1 root atlas  19G Jan  8  2007 E8coef-mod255
-rw-r--r-- 1 root atlas  19G Jan  6  2007 E8coef-mod256

-rw-r--r-- 1 root atlas  14G Jan  8  2007 E8mat-mod251
-rw-r--r-- 1 root atlas  14G Jan  6  2007 E8mat-mod253
-rw-r--r-- 1 root atlas  14G Jan  5  2007 E8mat-mod255
-rw-r--r-- 1 root atlas  14G Jan  6  2007 E8mat-mod256
```

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(Avec une impression normale, ces données couvriraient Lyon)

Some Statistics

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$152q^{22} + 3,472q^{21} + 38,791q^{20} + 293,021q^{19} + 1,370,892q^{18} + 4,067,059q^{17} + 7,964,012q^{16} + 11,159,003q^{15} + 11,808,808q^{14} + 9,859,915q^{13} + 6,778,956q^{12} + 3,964,369q^{11} + 2,015,441q^{10} + 906,567q^{9} + 363,611q^{8} + 129,820q^{7} + 41,239q^{6} + 11,426q^{5} + 2,677q^{4} + 492q^{3} + 61q^{2} + 3q$

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Value of this polynomial at q=1: 60,779,787

Some KLV polynomials

```
q^19+q^17+q^16+3q^15+3q^14+q^13+7q^12+5q^11+8q^10+7q^9+12q^8+10q^7+8q^6+10q^5+5q^4+4q^3+2q^2+2q+1
q^19+q^17+q^16+4q^15+2q^14+7q^13+11q^12+10q^11+5q^10+9q^9+8q^8+6q^7+2q^6+6q^5+5q^4+2q^3+q+1
q^19+q^17+q^16+4q^15+4q^14+12q^13+11q^12+13q^11+15q^10+12q^9+10q^8+9q^7+9q^6+7q^5+4q^4+q^3+q^2+q
q^20+2q^19+2q^18+q^17+2q^12+6q^11+10q^10+12q^9+12q^8+12q^7+11q^6+9q^5+7q^4+5q^3+3q^2+q
q^20+2q^19+2q^18+q^17+2q^12+6q^11+10q^10+12q^9+12q^8+13q^7+12q^6+9q^5+7q^4+5q^3+3q^2+q
q^20+2q^19+2q^18+q^17+2q^12+6q^11+10q^10+12q^9+13q^8+14q^7+12q^6+9q^5+7q^4+5q^3+3q^2+q
q^20+2q^19+2q^18+q^17+q^13+2q^12+3q^11+4q^10+4q^9+4q^8+4q^7+4q^6+2q^5+q^4+q^3+q^2+q
q^19+q^17+2q^16+2q^15+2q^14+2q^13+2q^12+4q^11+6q^10+10q^9+12q^8+11q^7+9q^6+7q^5+5q^4+2q^3+q^2+q
q^19+q^17+2q^16+2q^15+2q^14+2q^13+2q^12+4q^11+7q^10+11q^9+13q^8+11q^7+9q^6+7q^5+5q^4+2q^3+q^2+q
q^19+q^17+2q^16+2q^15+2q^14+3q^13+2q^12+2q^11+2q^10+3q^9+6q^8+6q^7+4q^6+6q^5+6q^4+3q^3+3q^2+2q+1
q^18+2q^17+4q^16+6q^15+7q^14+9q^13+11q^12+13q^11+14q^10+16q^9+16q^8+14q^7+10q^6+9q^5+6q^4+3q^3+2q^2+2q+1
q^18+2q^17+4q^16+6q^15+7q^14+9q^13+11q^12+13q^11+15q^10+17q^9+17q^8+14q^7+11q^6+9q^5+6q^4+3q^3+2q^2+2q+1
q^18+2q^17+4q^16+6q^15+7q^14+10q^13+12q^12+18q^11+22q^10+26q^9+26q^8+23q^7+19q^6+13q^5+9q^4+6q^3+3q^2+q
q^20+2q^19+2q^18+q^17+2q^12+6q^11+10q^10+13q^9+14q^8+14q^7+12q^6+9q^5+7q^4+5q^3+3q^2+q
q^19+q^17+2q^16+3q^15+4q^14+3q^13+2q^12+2q^11+q^10+q^9+q^8+3q^7+4q^6+6q^5+4q^4+5q^3+3q^2+q
q^19+q^17+2q^16+3q^15+4q^14+3q^13+8q^12+8q^11+7q^10+6q^9+8q^8+8q^7+2q^6+4q^5+3q^4+2q^3+q^2+1
q^19+q^17+2q^16+3q^15+4q^14+4q^13+10q^12+11q^11+13q^10+17q^9+18q^8+18q^7+15q^6+13q^5+8q^4+5q^3+4q^2+2q+1
q^19+q^17+2q^16+4q^15+2q^14+4q^13+2q^12+3q^11+3q^10+4q^9+4q^8+6q^7+2q^6+6q^5+3q^4+2q^3+q^2+2q+1
q^19+q^17+2q^16+4q^15+2q^14+6q^13+16q^12+13q^11+11q^10+17q^9+22q^8+14q^7+7q^6+13q^5+10q^4+3q^3+q^2+2q+1
q^19+q^17+2q^16+4q^15+4q^14+4q^13+5q^12+5q^11+6q^10+5q^9+7q^8+9q^7+11q^6+11q^5+9q^4+6q^3+3q^2+q
q^19+q^17+2q^16+5q^15+5q^14+6q^13+15q^12+15q^11+15q^10+14q^9+18q^8+11q^7+7q^6+10q^5+6q^4+3q^3+q^2+2q+1
q^25+q^21+q^18+2q^17+q^16+q^14+3q^13+q^12+q^11+3q^10+4q^9+q^8+q^6+q^5+q^2+q
q^25+q^21+q^18+2q^17+q^16+q^14+3q^13+q^12+q^11+4q^10+4q^9+q^8+2q^6+q^5+q^2+q
q^25+2q^21+2q^20+q^18+3q^17+3q^16+q^15+q^14+4q^13+4q^12+q^11+q^10+3q^9+3q^8+q^7+2q^5+2q^4+1
q^25+q^23+q^21+q^20+2q^19+2q^18+3q^17+2q^16+q^15+q^10+2q^9+3q^8+2q^7+2q^6+q^5+q^4+q^2+q
q^19+q^17+q^16+4q^15+2q^14+7q^13+11q^12+10q^11+5q^10+9q^9+8q^8+6q^7+2q^6+6q^5+5q^4+2q^3+q+1
q^18+2q^17+4q^16+6q^15+7q^14+9q^13+11q^12+13q^11+14q^10+15q^9+15q^8+13q^7+10q^6+8q^5+6q^4+3q^3+2q^2+2q+1
q^19+q^17+3q^16+q^15+2q^14+5q^13+6q^12+4q^11+3q^10+7q^9+7q^8+2q^7+2q^6+4q^5+3q^4+q
```

Fokko du Cloux
December 20, 1954 - November 10, 2006

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We now want to use this data to answer some questions, for any Lie group G:

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5. What does this tell us about number theory and automorphic forms?

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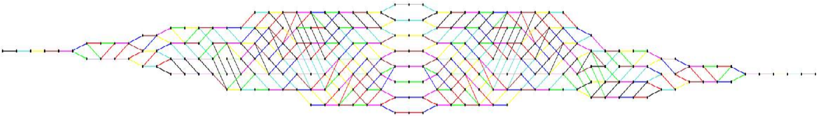

www.liegroups.org