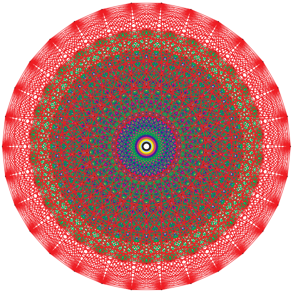
The Atlas of Lie Groups and Representations

www.liegroups.org



Atlas Project Members

- Jeffrey Adams
- Dan Barbasch
- Birne Binegar
- Bill Casselman
- Dan Ciubotaru
- Scott Crofts
- Fokko du Cloux
- Alfred Noel
- Tatiana Howard
- Alessandra Pantano
- Annegret Paul
- Patrick Polo
- Siddhartha Sahi
- Susana Salamanca
- John Stembridge
- Peter Trapa
- Marc van Leeuwen
- David Vogan
- Wai-Ling Yee
- Jiu-Kang Yu
- Gregg Zuckerman

red: directly worked on the E_8 calculation

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- New York Times Science Section (March 20)
- Science
- Nature (online)
- Le Monde
- London Times
- Los Angeles Times
- Scientific American (online)
- Al Arabiya TV (satellite, Dubai)
- Economist
- Yahoo news (top 5 news, top emailed news story for several days)
- Good Morning America
- Fox News
- NPR
- Front page of the NSF site
- AP and other wire services

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- "100 year old problem"

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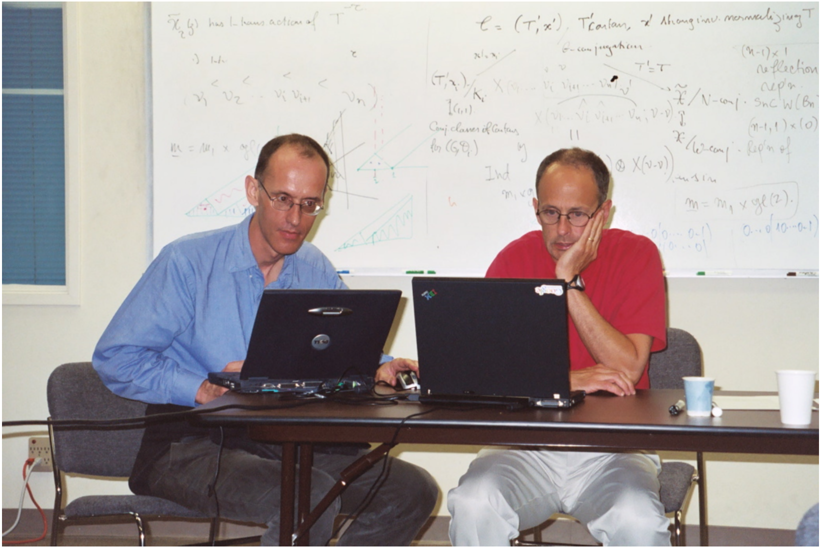
- Connection with string theory
- It was not necessary to overly simplify the material or invent ties to other branches of mathematics or science


reflection
Conj classes of Cartans
Ind

What Fokko did

Abstract Mathematics		
Harish-Chandra		
Langlands		**Algorithm**
Knapp/Zuckerman/Vogan	$\longrightarrow$	Combinatorial set
Vogan		
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What Fokko did

Abstract Mathematics		**Algorithm**		**Software**
Harish-Chandra				
Langlands				
Knapp/Zuckerman/Vogan	$\longrightarrow$	Combinatorial set	$\longrightarrow$	C++ code
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Fokko du Cloux

December 20, 1954–November 10, 2006

Overview of the Atlas Project

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A representation π of G is a homomorphism $\pi : G \to GL(\mathcal{H})$ (invertible operators on a Hilbert space $\mathcal{H}$). It is unitary if it is length preserving: $|\pi(g)v| = |v|$ for all $v \in \mathcal{H}$. It is irreducible if there are no closed invariant subspaces.

Example: $\mathcal{H} = L^2(G)$, $\pi(g)(f)(x) = f(g^{-1}x)$ This is the regular representation. It is highly reducible:

$$L^2(G) \simeq \int_{\hat{G}} \pi d\mu(\pi)$$

where $d\mu(\pi)$ is a measure on the space $G\hat{}$ of irreducible unitary representations of G.
More generally if G acts on X, preserving a measure μ, study action of G on X by linearizing, i.e. study representation of G on $L^2(X)$.

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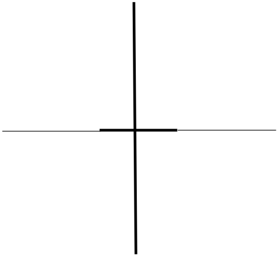
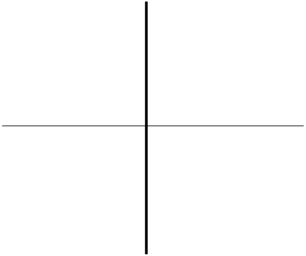
- $GL(n,\mathbb{R})$ (Vogan, 1986)
- real rank 1: $SU(n,1), SO(n,1), Sp(n,1)$
- Complex classical groups: $SL(n,\mathbb{C}), SO(n,\mathbb{C}), Sp(2n,\mathbb{C})$ (Barbasch, 1989)

A few other small cases, no other infinite families

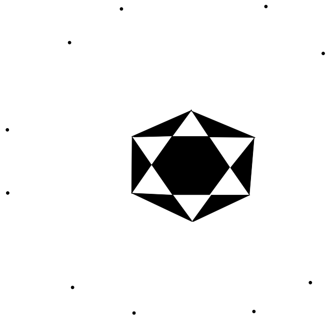
Unitary dual of $SL(2,\mathbb{R})$





$\mathbb{Z} - 0$

Spherical unitary dual of G_2



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Atlas of Lie Groups and Representations:

Take this idea seriously!

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    1. Provide software to compute with Lie groups and their representations.
    2. Provide information and interactive tools on a web site for non-experts.

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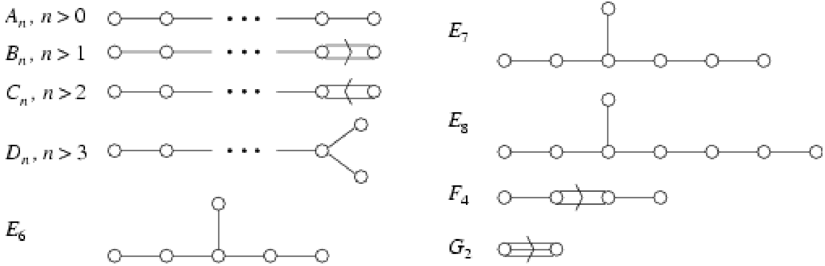
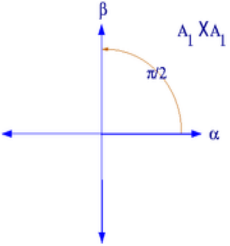
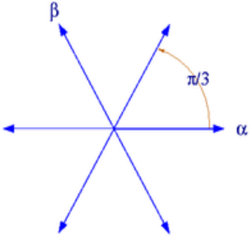
1. Irreducible root systems
2. Irreducible Dynkin diagrams
3. Simple complex Lie algebras
4. Simple complex Lie groups
5. A_n, B_n, C_n, D_n, $n = 1, 2, 3, \dots$ (classical) G_2, F_4, E_6, E_7, E_8 (exceptional)

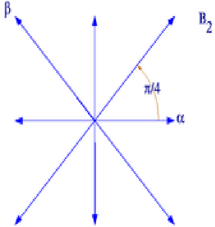
Dynkin Diagrams

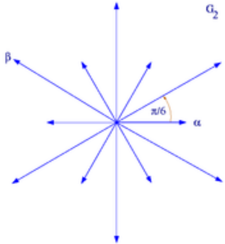


Rank Two Root Systems









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$\{(g,h) \in GL(n,\mathbb{C}) \times GL(m,\mathbb{C}) \mid \det(g)\det(h) = 1\}$

Grothendieck classified complex reductive (algebraic) groups in terms of root data:

$$(X, \Phi, X^\vee, \Phi^\vee)$$

where $X, X^\vee \simeq \mathbb{Z}^n$, Φ and $\Phi^\vee$ are finite subsets of $X, X^\vee$, in bijection ($\alpha \to \alpha^\vee$), satisfying properties:
$\langle \alpha, \alpha^\vee \rangle \in \mathbb{Z}$
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Data: two $m \times n$ matrices of integers.

Beautifully suited to a computer!

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There is always a unique compact real form $(SU(n), SO(n))$

There is always a unique split real form $(SL(n, \mathbb{R}), SO(n, n))$

First goal: write software to input an arbitrary real reductive group, and compute its structure theory.

The unitary representations occuring in $L^2(G)$ are known (Harish-Chandra, 1970s). These are called tempered: $\widehat{G}_t \subset \widehat{G}_u$.

Unitary representations are contained in a larger class, called admissible: $\widehat{G}_u \subset \widehat{G}_a$. These are also known (Langlands, Knapp, Zuckerman, Vogan)

$$\widehat{G}_t \subset \widehat{G}_u \subset \widehat{G}_a$$

To compute $\widehat{G}_u$: take each representation $\pi \in \widehat{G}_a$, and test if it is unitary. Not obvious this is a finite calculation even for a single π (not to mention uncountably many π).

Finite Calculation

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We will always work in the set of representations with the same “central character” as the trivial representation. This is the hardest case, others reduce to this.

Second Goal: find an algorithm to compute $\widehat{G}_a$, and write software to implement it.

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Although the mathematics is “known”, we greatly deepened our understanding of the mathematics in doing this.
For example: figuring out the data structures to adequately capture the mathematics required us to rethink the mathematics carefully.

Old days: representation of G on $L^2(X)$ (for example)

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Example: $G = SL(2,\mathbb{R})$ on $L^2(\mathbb{R})$:

$$\pi_\nu(g)f(x) = |-bx+d|^\nu f((ax-c)/(-bx+d))$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

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We parametrize $\widehat{G}_{a,1}$ by a finite set $\mathcal{X}$. Throw away π, and keep only the parameter space $\mathcal{X}$.

Algorithm to compute $\widehat{G}_{a,1}$

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Problem: Compute the orbits of H_m on X. This is a finite set. Compute the closure relations.

Combinatorial Solution:

$\widetilde{W}$ = generalized permutation matrices (one non-zero entry in each row and column)
$\simeq S^n \rtimes \mathbb{C}^{\times n}$
D = diagonal matrices
$\mathcal{X} = \{x \in \tilde{W} \mid x^2 = 1\}/D$

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Fact: $\mathcal{X}$ is in natural bijection with $\cup_m X/H_m$
Computing $\mathcal{X}$ is an explicit combinatorial problem in finite group theory, a little harder than computing the elements of order 2 in S^n.

The software now calculates $\widehat{G}_{a,1}$ for any G.

Example: $SL(2,\mathbb{R})$:

```
This is the Atlas of Reductive Lie Groups Software Package version 0.2.5.
Build date: Nov 24 2006 at 09:16:16.
Enter "help" if you need assistance.

empty: block
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (hit return for stdout):
0(0,1):  1    (2,*)    [i1]  0
1(1,1):  0    (2,*)    [i1]  0
2(2,0):  2    (*,*)    [r1]  1  1
```

$Sp(4,\mathbb{R})$:

```
 0( 0,6):   1   2     ( 6, *)  ( 4, *)     [i1,i1]  0
 1( 1,6):   0   3     ( 6, *)  ( 5, *)     [i1,i1]  0
 2( 2,6):   2   0     ( *, *)  ( 4, *)     [ic,i1]  0
 3( 3,6):   3   1     ( *, *)  ( 5, *)     [ic,i1]  0
 4( 4,4):   8   4     ( *, *)  ( *, *)     [C+,r1]  1  2
 5( 5,4):   9   5     ( *, *)  ( *, *)     [C+,r1]  1  2
 6( 6,5):   6   7     ( *, *)  ( *, *)     [r1,C+]  1  1
 7( 7,2):   7   6     (10,11)  ( *, *)     [i2,C-]  2  2,1,2
 8( 8,3):   4   9     ( *, *)  (10, *)     [C-,i1]  2  1,2,1
 9( 9,3):   5   8     ( *, *)  (10, *)     [C-,i1]  2  1,2,1
10(10,0):  11  10     ( *, *)  ( *, *)     [r2,r1]  3  1,2,1,2
11(10,1):  10  11     ( *, *)  ( *, *)     [r2,rn]  3  1,2,1,2
```

So far we've said the atlas software should (and does) do:

1. Calculate with structure theory of reductive groups
2. Calculate the admissible dual $\widehat{G}_{a,\mathbf{1}}$.

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One more ingredient is needed.

Character theory

Let G be a finite group. Then a representation $\pi : G \to GL(n, \mathbb{C})$ is determined by its character $\theta_\pi(g) = \mathrm{Trace}(\pi(g))$.
The functions θ_π are a basis of $L^2(G)^G$.
So are $\chi_{\mathcal{O}}$ where $\mathcal{O}$ is a conjugacy class.

The character table of G contains all information about its representations:

Character Table of Weyl Group of type D4

Class		1	2	3	4	5	6	7	8	9	10	11	12	13
Size		1	1	6	6	6	12	12	32	12	24	24	24	32
Order		1	2	2	2	2	2	2	3	4	4	4	4	6
p =	2	1	1	1	1	1	1	1	8	2	5	4	3	8
p =	3	1	2	3	4	5	6	7	1	9	10	11	12	2
X.1	+	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	+	1	1	1	1	1	-1	-1	1	1	-1	-1	-1	1
X.3	+	2	2	2	2	2	0	0	-1	2	0	0	0	-1
X.4	+	3	3	-1	-1	3	-1	-1	0	-1	-1	1	1	0
X.5	+	3	3	3	-1	-1	-1	-1	0	-1	1	1	-1	0
X.6	+	3	3	-1	3	-1	-1	-1	0	-1	1	-1	1	0
X.7	+	3	3	-1	-1	3	1	1	0	-1	1	-1	-1	0
X.8	+	3	3	3	-1	-1	1	1	0	-1	-1	-1	1	0
X.9	+	3	3	-1	3	-1	1	1	0	-1	-1	1	-1	0
X.10	+	4	-4	0	0	0	-2	2	1	0	0	0	0	-1
X.11	+	4	-4	0	0	0	2	-2	1	0	0	0	0	-1
X.12	+	6	6	-2	-2	-2	0	0	0	2	0	0	0	0
X.13	+	8	-8	0	0	0	0	0	-1	0	0	0	0	1

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$$I(x) = \sum_{y \in \mathcal{X}} m(x,y)\pi(y) \quad m(x,y) \in \mathbb{Z}$$

Langlands, Zuckerman: this identity is invertible:

$$\pi(x) = \sum M(x,y) I(y)$$

This gives a character formula for $\pi(x)$.

Kazdhan-Lusztig, Vogan:
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Problem: compute $P_{x,y}$.

Summary of the atlas software

The atlas software now does the following:

1. Input arbitrary reductive complex algebraic group $G(\mathbb{C})$
2. Input real form G of $G(\mathbb{C})$
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We hope this will be enough information to compute the unitary dual of G. It *is* enough information to list the most interesting, conjecturally unitary representations: the unipotent representations of Jim Arthur.

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Split Group	time in seconds
$SL(2,\mathbb{R})$	.003
G_2	.008
F_4	.13
A_8	.17
A_9	.8
E_6	1.3
A_{10}	15
E_7	107
E_8	∞

Overview of the E_8 calculation

Recall E_8 is the largest exceptional group. The split real form is a real manifold of dimension 248, and it has $453,060$ irreducible representation in $\widehat{G}_{a,1}$.

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This is an upper triangular matrix, of size $453,060$, with 1^s on the diagonal, and polynomial entries. Each polynomial has degree ≤ 31.

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5. To force us to improve the technology. We have much harder calculations to do to compute $\widehat{G}_u$. We have no hope of computing the unitary dual of F_4 if we can't compute KLV polynomials for E_8. It would not be enough to find a big enough computer.
6. Because E_8 is a particularly interesting group, and arises in string theory.

Recursion Relations

$\mathcal{X}$ is the set of parameters.
There is a partial order $<$ on $\mathcal{X}$, and a length function. For E_8 $\ell(x) \leq 62$.
The matrix is upper triangular:
$P_{x,x} = 1$
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Long list of complicated recursion formulas.

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Type I: There exists y' with $\ell(y') < \ell(y)$ such that

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Type II: There is y', $\ell(y') = \ell(y)$, y'', $\ell(y'') = \ell(y) - 1$,

$$P_{x,y} = \sum_{\ell(x')=\ell(x)+1} P_{x',y'} + \sum_{x''} P_{x'',y''} \quad (\leq 4 \text{ terms})$$

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We need to keep all $P_{x,y}$ in RAM!

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Each polynomial has ≤ 32 coefficients.

$450,060^2 \times 32 = 6.5$ trillion coefficients =26 trillion bytes

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Store only the distinct polynomials.

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$6 \times 10^9 \times 32 = 200$ billion coefficents, or 800 billion bytes
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David threaded the code to run many calculations simultaneously (on some platforms this slowed the calculation down

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This gets us down to about $15 + 4 = 19$ billion bytes

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In fact we can work $\pmod{n}$ for any n.

The final result

In the end:

Run the program 4 times modulo $n = 251, 253, 255, 256$
Least common multiple: 4,145,475,840
Combine the answers using the Chinese Remainder Theorem.
Answer is correct if the biggest coefficient is less then 4,145,475,840
Total time (on sage): 77 hours

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What comes next?

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Check back in a few years…