# Atlas Meeting V, Palo Alto, July 16-20, 2007

## David Vogan: Associated varieties [(1)]

Wednesday 13:30

Let X be an irreducible HC module. How can you compute AV(X), which is a union of components of $\overline{\mathscr{O}} \cap (\mathfrak{g}/\mathfrak{k})^*$?

Recall $\overline{\mathscr{O}} = \mathrm{AV}(\mathrm{Ann}\,\mathrm{X})$. Recall that the dimension of a Richardson orbit for the parabolic $\mathfrak{q}$ is

$$\#\mathrm{R}(\mathfrak{g}) - \#\mathrm{R}(\mathfrak{l})$$

where $\mathfrak{l}$ is the Levi of $\mathfrak{q}$.

Last night it was explained that $\overline{\mathscr{O}}$ is contained in the closure of the Richardson orbit for any parabolic $\mathfrak{q}$ corresponding to the descent set (= $\tau$-invariant) of some Y in the cell of X. That is, $\overline{\mathscr{O}}$ is contained in the intersection of all closures of Richardson orbits obtained that way. This gives an *upper* bound for $\overline{\mathscr{O}}$.

Question: is it true that any special orbit is an intersection of Richardson orbits? Monty replies: yes for classical types, no for exceptional ones.

Recall that there is the Spaltenstein duality (order-reversing) $d$ between special nilpotent orbits in ${}^\vee\mathfrak{g}$ and those in $\mathfrak{g}^*$. Then $d(\mathscr{O})$ is contained in the closure of Richardson orbit for ${}^\vee\mathfrak{q}$, which is the parabolic corresponding to the *complement* of $\tau(\mathrm{Y})$.

Thus, $\overline{\mathscr{O}}$ contains $d$ of the Richardson orbit for ${}^\vee\mathfrak{q}$. This gives a *lower* bound for $\overline{\mathscr{O}}$.

Now, try to get at AV(X).

1) Look in your cell to see if you can find a representation $\mathrm{A}_\mathfrak{q}$ (Zuckerman derived functor). The blocku command gives precisely these, which are exactly the unitary ones in the case of infinitesimal character $\rho$.

The Levi of $\mathfrak{q}$ corresponds to the non-$*$ simple roots in the blocku output.

If you find one, then $\mathrm{AV(X)} = \mathrm{AV}(\mathrm{A}_\mathfrak{q})$ and by Borho-Brylinski the latter is

$$\mathrm{K}\cdot(\mathfrak{g}/\mathfrak{k}+\mathfrak{q})^* = \mathrm{K}\cdot(\mathfrak{q}^\perp \cap \mathfrak{k}^\perp) \cong \mathrm{K}\cdot(\mathfrak{u}\cap\mathfrak{p}) \subseteq \mathfrak{g}$$

(using the not-so-good identification $\mathfrak{g}^* \cong \mathfrak{g}$). This is only sort of explicit, that is, not enough for some practical purposes.

Consider now generalized $\mathrm{A}_\mathfrak{q}$'s.

Start with an $\mathrm{A}_\mathfrak{q}$. This corresponds to an output $\#m$ in the blocku command.

$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ corresponds to non-$*$ simple roots. Move from $\#m$ (inside the block) by performing cross actions and inverse Cayley transforms, using only roots in L.

(1) notes by Patrick Polo

This leads to a family $\mathscr{F}(\#m)$ of representations of $\mathfrak{g}$; it corresponds bijectively to the block of the trivial module for L. Take some $\#k$ in $\mathscr{F}(\#m)$; it corresponds to a representation Z in the trivial block of L, and the G-representation $\#k$ equals

$$\mathscr{R}(\mathrm{Z})$$

where $\mathscr{R}$ is the Zuckerman functor. Then one has (identifying $\mathfrak{g}^* = \mathfrak{g}$ and similarly for $\mathfrak{l}$):

$$\mathrm{AV}(\#k) = \mathrm{K} \cdot (\mathrm{AV}(\mathrm{Z}) + \mathfrak{u} \cap \mathfrak{p})$$

where AV(Z) is contained in $(\mathfrak{l} \cap \mathfrak{p})_{\mathrm{nilp}}$. Thus calculating AV($\#k$) is reduced to the lower rank group L. The G representations in $\mathscr{F}(\#m)$ are the so called generalized $\mathrm{A}_{\mathfrak{q}}$. This provides a larger library of modules that one can sort of understand.

For $\mathrm{E}_8(\mathbb{R})$, any nilpotent complex orbit has at most 3 real forms. Recall that the number of real forms of the orbit is the number of K-orbits in $\mathscr{O} \cap (\mathfrak{g}/\mathfrak{k})^*$; this is also the number of irreducible components of

$$\overline{\mathscr{O}} \cap (\mathfrak{g}/\mathfrak{k})^*.$$

Since the associated variety AV(X) is the union of some of these components, this gives the upper bound 3 for the number of irreducible components of AV(X).

Let $\mathrm{K}e_i \cong \mathrm{K}/\mathrm{H}_i$ be the components of AV(X). Then X gives rise to a "genuine virtual algebraic representation" $\tau_i$ of $\mathrm{H}_i$.

X is part of a coherent family X($\lambda$) and one has

$$\dim \eta_i(\lambda) = \mathrm{P}_i(\lambda),$$

where $\mathrm{P}_i$ is a polynomial, namely a multiple of Joseph's Goldie rank polynomial associated with the primitive ideal which is the annihilator of X.

Here is an incomplete sketch of one more technique to compute an associated variety, which can give a *reducible* answer. This did not appear in the lecture. Suppose $\mathscr{O}$ is an even complex nilpotent orbit, with Jacobson-Morozov parabolic subalgebra $\mathfrak{q}$ (with Levi factor corresponding to the zeros in the Dynkin diagram of $\mathscr{O}$). In this case $\mathscr{O}$ is necessarily the Richardson orbit defined by $\mathfrak{q}$. Suppose that $\mathrm{Q}(\mathbb{R}) \subset \mathrm{G}(\mathbb{R})$ is a real parabolic subgroup conjugate to $\mathfrak{q}$. (Such a parabolic exists if and only if $\mathscr{O}$ has real forms for $\mathrm{G}(\mathbb{R})$.) Now let $\xi(\mathbb{R})$ be any unitary finite-dimensional character of $\mathrm{Q}(\mathbb{R})$, and consider the unitary degenerate principal series representation

$$\mathrm{I}(\xi(\mathbb{R})) = \mathrm{Ind}_{\mathrm{Q}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}(\xi(\mathbb{R})).$$

***Theorem 1***. — *In the setting above,* $\mathrm{AV}(\mathrm{I}(\xi(\mathbb{R}))$ *is equal to* $\overline{\mathscr{O}} \cap (\mathfrak{g}/\mathfrak{k})^*$. *Each irreducible component of this intersection occurs in the associated variety of exactly one irreducible summand of* $\mathrm{I}(\xi(\mathbb{R}))$. *Conversely, the associated variety of each summand is a union of irreducible components of the intersection.*

In case $\mathrm{I}(\xi(\mathbb{R}))$ has integral infinitesimal character, its decomposition into irreducibles can be computed by atlas. All the summands belong to translation families in a single block. If there is a single summand, then its associated variety corresponds to all the real forms of $\mathscr{O}$. (This occurs for $\mathscr{O}$ the principal orbit in $\mathrm{SL}(2)$ and $\xi(\mathbb{R})$ trivial. The corresponding representation is the unique element of the $\mathrm{SL}(2,\mathbb{R})$ block dual to $\mathrm{SU}(2)$. Its associated variety therefore has two components.)

If there are several summands, one can hope by some other technique to compute the associated varieties of some of them. The remaining summands must then partition the remaining components of $\overline{\mathscr{O}} \cap (\mathfrak{g}/\mathfrak{k})^*$. I do not have an example of this offhand.