Atlas Meeting V, Palo Alto, July 16-20, 2007

Monty McGovern: Nilpotent orbits [(1)]

Monday 14:30

Start by an advertisement for Collingwood-McGovern, Nilpotent orbits in Semisimple Lie algebras, Chapman & Hall (unfortunately very expensive: \$ 160), and a volume in the series Springer Encyclopedia of Math. Sciences, Transformation Groups & Invariant Theory, with articles by Carrell, Bialynicki-Birula, and McGovern (slightly less expensive: \$125).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We say $x \in \mathfrak{g}$ is nilpotent (resp. semisimple) if $\operatorname{ad} x$ acts nilpotently (resp. semisimply) on $\mathfrak{g}$.

One justification is that if $\mathfrak{g} = \mathfrak{sl}_n$, x is nilpotent (resp. semisimple) iff x is so as a matrix.

If $\mathrm{G} = \mathrm{Ad}(\mathfrak{g})$, then x is nilpotent iff every element of $\mathrm{G}x$ is nilpotent, and similarly for semisimple, so we may apply these notions to G-orbits.

Semisimple orbits Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Every semisimple orbit meets $\mathfrak{h}$ exactly in one orbit of W, the Weyl group, and

$$(\mathfrak{g}/\mathrm{G})_{\mathrm{ss}} \cong \mathfrak{h}/\mathrm{W}.$$

Nilpotent orbits First of all, if $e \in \mathfrak{g}$ is nilpotent, then it is part of a standard $\mathfrak{sl}_2$-triple (e, f, h). This is not uniquely determined by e, but $\mathrm{G}h$ is uniquely determined by $\mathrm{G}e$, so we get a well-defined map from nilpotent orbits to semisimple orbits.

We find that all eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}$ are integral. We can find a Cartan subalgebra $\mathfrak{h}$ containing h and such that we can choose simple roots for $\mathfrak{h}$ in $\mathfrak{g}$ such that all eigenvalues of $\operatorname{ad} h$ on simple root spaces $\mathfrak{g}_\alpha$ are 0, 1, or 2. Thus we obtain a well-defined *weighted Dynkin diagram*: each node in the ordinary Dynkin diagram is labelled by the eigenvalue of $\operatorname{ad} h$ on the corresponding simple root space $\mathfrak{g}_\alpha$.

By no means do all possible weighted diagram occur!

– the diagram with all 0s corresponds to $\{0\}$ orbit

– the diagram with all 2s corresponds to the principal nilpotent orbit (the unique open dense orbit in the nilpotent cone)

Dynkin also considered the question: given e, which reductive subalgebras of $\mathfrak{g}$ contain e? In fact, he restricted to what he called "regular reductive subalgebras" (= containing a fixed Cartan subalgebra $\mathfrak{h}$) and meeting $\mathrm{G}e$. He labelled the orbit by "the smallest" reductive subalgebra with this property (not a unique one, choices to be made here).

[(1)] notes by Patrick Polo

Bala-Carter classification. They restricted attention to Levi factors of parabolic subalgebras; they found that for any nilpotent orbit $\mathrm{G}e$, there is a *unique* (up to conjugacy) minimal Levi factor $\mathfrak{l}$ meeting $\mathrm{G}e$, and it meets $\mathrm{G}e$ in a single orbit $\mathrm{L}e'$.

Given $\mathfrak{l}$, they asked which orbits meet $\mathfrak{l}$ but no smaller Levi factor, calling them *distinguished in* $\mathfrak{l}$. They found the distinguished orbits are all even (that is, only 0s and 2s) and are all Richardson (that is, meet the nilradical of the parabolic in a dense set).

Finally, they observed that for a parabolic subalgebra $\mathfrak{p}$ with nilradical $\mathfrak{n}$, Richardson orbits meeting densely, we can decide whether or not this orbit is distinguished by a root system calculation.

Upshot, for the exceptional cases: 5, resp. 16, 21, 45, 70 for G_2, resp. F_4, E_6, E_7, E_8, of which 2, resp. 4, 3, 6, 11 are distinguished (that is, distinguished in $\mathfrak{g}$).

Classification: except for technicalities involving conjugacy classes of Levi factors, we can label any nilpotent orbit by a pair of Cartan types, one corresponding to a Levi factor, the second to a subalgebra of it.

In the classical cases, we can bypass all these considerations and appeal to the Jordan canonical form.

– $\mathfrak{g} = \mathfrak{sl}_n$: nilpotent orbits $\leftrightarrow$ partitions of n

– $\mathfrak{g} = \mathfrak{so}_n$ (n even or odd): nilpotent orbits $\leftrightarrow$ partitions of n in which even parts occur with even multiplicities, except that if all parts are even, we get *two* orbits attached to the partition, labelled I and II

– $\mathfrak{g} = \mathfrak{sp}_{2n}$: $\leftrightarrow$ partitions of $2n$ in which odd parts occur with even multiplicities.

Alternatively, more Atlasly, one can produce nilpotent orbits from Weyl group elements as follows.

Fix a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, let $\mathrm{W} = \mathrm{W}(\mathfrak{g}, \mathfrak{h})$. Given $w \in \mathrm{W}$, look at

$$\mathfrak{n} \bigcap w(\mathfrak{n});$$

a unique nilpotent orbit will meet this in an open dense set. Call this orbit $\mathscr{O}_w$. Using Bala-Carter, Steinberg proved in a very clever way that all nilpotent orbits occur in that way. One has $\mathscr{O}_w = \mathscr{O}_{w^{-1}}$, but there is much more collapsing. E.g. in E_8, according to David there is about 70 thousands of involutions, but only 70 nilpotent orbits.

Moreover, one can also study irreducible components of $\overline{\mathscr{O}} \cap \mathfrak{n}$, called *orbital varieties*. They all take the form:

$$\overline{\mathrm{B} \cdot (\mathfrak{n} \cap w\mathfrak{n})}$$

for some w such that $\mathscr{O}$ meets $\mathfrak{n} \cap w\mathfrak{n}$ densely.

Real case Nilpotent $K_{\mathbb{C}}$-orbits in $\mathfrak{p}$ can be studied in much the same way, using standard triples (h, e, f) where $h \in \mathfrak{k}$ and $e, f \in \mathfrak{p}$. The weighted Dynkin diagrams occuring in this fashion were classified by Djoković in two papers in J. Algebra 112 and 116 (1988).