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On the unitary dual of Hecke algebras with unequal parameters

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1. The graded Hecke algebra and Langlands classification

Let $(\mathcal{X}, \mathcal{R}, \check{\mathcal{X}}, \check{\mathcal{R}}, \Pi)$ be a (reduced) root datum with Weyl group W. Let $c : \Pi \to \mathbb{N}$ be a parameter set, i.e., a function such that $c(\alpha_j) = c(\alpha_i)$ whenever α_j and α_i are W-conjugate. Set $\mathfrak{t} = \check{\mathcal{X}} \otimes \mathbb{C}$, $\mathfrak{t}^* = \mathcal{X} \otimes \mathbb{C}$, and similarly $\mathfrak{t}_{\mathbb{R}}$ and $\mathfrak{t}^*_{\mathbb{R}}$.

Definition 1.1. *The graded Hecke algebra $\mathbb{H}$ is the vector space $\mathbb{C}W \otimes \mathbb{A}$, where $\mathbb{A} = S(\mathfrak{t}^*)$ subject to the commutation relation*

$$s_\alpha \cdot \omega = s_\alpha(\omega) \cdot s_\alpha + c(\alpha)\omega(\check{\alpha}), \qquad \textit{for } \alpha \in \Pi, \omega \in \mathfrak{t}^*.$$

If V is a finite dimensional irreducible module, $\mathbb{A}$ induces a generalized weight space decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{t}} V_\lambda.$$

Call λ a *weight* if $V_\lambda \neq 0$.

Definition 1.2. *The irreducible module V is called essentially tempered if $Re(\omega_i(\lambda)) \leq 0$, for all weights $\lambda \in \mathfrak{t}$ of V and all fundamental weights $\omega_i \in \mathfrak{t}^*$. If in addition $Re(x(\lambda)) = 0$, for all $x \in \mathcal{X}$ perpendicular on $\check{\mathcal{R}}$, V is called tempered. If V is tempered and $Re(\omega(\lambda)) < 0$, for all λ, ω_i as above, V is called a discrete series.*

Example. One discrete series which appears in every Hecke algebra is the *Steinberg module St*. As a W-representation, $St|_W = sgn$, and the unique $\mathbb{A}$-weight is $-\sum c(\alpha_i)\check{\omega}_i$, where $\check{\omega}_i \in \mathfrak{t}$ are the fundamental coweights.

The center of the Hecke algebra $\mathbb{A}^W$ acts by a character on V. The *central characters* are therefore parametrized by W-orbits on $\mathfrak{t}$. Call a central character *real* if the corresponding W-orbit is in $\mathfrak{t}_{\mathbb{R}}$.

We recall the Langlands classification due to Evens. For every $\Pi_P \subset \Pi$, one can define $\mathcal{R}_P$ and $\check{\mathcal{R}}_P$. Let $\mathbb{H}_P$ be the Hecke algebra attached to the root datum $(\mathcal{X}, \mathcal{R}_P, \check{\mathcal{X}}, \check{\mathcal{R}}_P, \Pi_P)$. This can be regarded as a subalgebra of $\mathbb{H}$. Moreover, $\mathbb{H}_P$ decomposes as

$$\mathbb{H}_P = \mathbb{H}_M \otimes S(\mathfrak{a}^*),$$

where $\mathfrak{a} = \{\nu \in \mathfrak{t} : \alpha(\nu) = 0, \text{ for all } \alpha \in \Pi_P\}$, and $\mathbb{H}_M$ is the Hecke algebra attached to $(\mathcal{X}', \mathcal{R}_P, \check{\mathcal{X}}', \check{\mathcal{R}}_P, \Pi_P)$, where $\mathcal{X}' \subset \mathcal{X}$ and $\check{\mathcal{X}}' \subset \check{\mathcal{X}}'$ are the subsets perpendicular to $\mathfrak{a}$, respectively $\mathfrak{a}'$.

Theorem 1.3 (Evens)**.** *Every irreducible $\mathbb{H}$-module appears as the unique irreducible quotient $L(P,V,\nu)$ of an $I(P,V,\nu) = \mathbb{H} \otimes_{\mathbb{H}_P} (V \otimes \mathbb{C}_\nu)$, where V is tempered for $\mathbb{H}_M$, $\nu \in \mathfrak{a}^+ = \{\nu \in \mathfrak{a} : \alpha(\nu) > 0,$ for all $\alpha \in \Pi_P\}$.*
If $L(P,V,\nu) \cong L(P',V',\nu')$, then $P = P'$, $V \cong V'$, and $\nu = \nu'$.

Let w_0 denote the longest element in W. The Hecke algebra $\mathbb{H}$ has a $*$-operation:

$$w^* = w^{-1},\ w \in W;$$
$$\omega^* = -w_0 \cdot (w_0\overline{\omega}) \cdot w_0,\ \omega \in \mathfrak{t}^*.$$

Then one can define Hermitian and unitary modules for $\mathbb{H}$.

Problem. *Identify the unitary dual of $\mathbb{H}$. It is sufficient to determine the unitary modules with real central character.*

The tempered modules are classified for many parameter sets c, but not for all. We remark that however, results of Opdam show that all tempered modules are unitary. (The Hecke algebra $\mathbb{H}$ carries a inner product defined using $*$, so one can construct the Hilbert space completion of $\mathbb{H}$, call it $\mathfrak{H}$. Then one shows that all discrete series are submodules of $\mathfrak{H}$.)

2. Geometric graded Hecke algebras

The Hecke algebra with equal parameters $\mathbb{H}_0$ arises from the theory of Iwahori-spherical representations of a split p-adic group. Its dual was classified by Kazhdan and Lusztig.

Let G be a simply connected quasi-simple *complex* group with Lie algebra $\mathfrak{g}$, and let $\mathcal{N}_G$ denote the set $\mathcal{N}_G = \{(e,\phi) : e$ nilpotent in $\mathfrak{g},\ \phi \in \widehat{A(e)}\}$. (Here, $A(e)$ denotes the component group of the centralizer of e in G.) Recall that there exists an injection

$$\widehat{W} \hookrightarrow \mathcal{N}_G,$$

the Springer correspondence. Then

$$\widehat{\mathbb{H}_0} \leftrightarrow \{(s,e,\psi) : s \in \mathfrak{g} \text{ semisimple},\ e \in \mathfrak{g} \text{ nilpotent},\ [s,e] = 2e,$$
$$\psi \in \widehat{A(s,e)} \text{ satisfying some restrictions}\}/G.$$

The restrictions are that $\psi|_{Z(G)} = 1$, and that ψ "come" from Springer's correspondence.

Many other Hecke algebras (with unequal parameters) appear when one describes other classes of representations (*unipotent*) of p-adic groups. We recall some results of Lusztig for the geometric classification of such algebras.

Firstly, the *generalized Springer correspondence* gives a bijection

$$\mathcal{N}_G \leftrightarrow \sqcup_{j\geq 0}\widehat{W_j},$$

where W_j are Weyl groups, and $W_0 = W$. For example, if $G = Sp(6)$, $|\mathcal{N}_G|$ has 16 elements, and the bijection is $\mathcal{N}_G \leftrightarrow \widehat{W(C_3)} \sqcup \widehat{W(C_2)} \sqcup \xi$. (Here ξ denotes a single representation.)

Then Lusztig's classification is:

$$\sqcup_{j\geq 0}\widehat{\mathbb{H}_j} \leftrightarrow \{(s,e,\psi) : \psi \text{ is unrestricted}\}/G.$$

In the $sp(6)$ example, the left hand side is the union of dual for $\mathbb{H}(C_3, 2, 2)$, $\mathbb{H}(C_2, 2, 3)$, and one cuspidal representation parametrized by the nilpotent (42) in $\mathfrak{sp}(6)$.

Each $\mathbb{H}_j$ and W_j are attached to a triple $(M_j, \mathcal{O}_j, \mathcal{L}_j)$, where M_j is a Levi subgroup with Lie algebra $\mathfrak{m}_j$, $\mathcal{O}_j$ is a nilpotent orbit in $\mathfrak{m}_j$, and $\mathcal{L}_j$ is a *cuspidal local system* on $\mathcal{O}_j$. The cuspidal local systems are those parametrizing representation ξ as above (those which do not belong to a smaller Weyl group) in the generalized Springer correspondence. (The notion is very restrictive, for example in $\mathfrak{sp}(2n)$ the only nilpotent orbits carrying a cuspidal local system are of the form $(2, 4, \ldots, 2k)$.) Set $W_j = N_G(M_j)/M_j$. This is a Coxeter group. The Hecke algebra is defined using $\mathfrak{t}_j = Z(\mathfrak{m}_j)$, $\mathcal{R}_j \subset \mathfrak{t}_j^*$ the nonzero weights of $\mathfrak{t}_j$ on $\mathfrak{g}$. The simple roots Π_j and the parameter set $c : \Pi_j \to \mathbb{N}$ are obtained from embedding $\mathfrak{m}_j$ maximally into Levi subalgebras $\mathfrak{m}'_j$.

Definition 2.1. *For these Hecke algebras, a geometric parameter (s, e, ψ) is called tempered if $\{Re(s), e\}$ can be embedded into a Lie triple of $\mathfrak{g}$. (Here $Re(s)$ denotes the hyperbolic part of the semisimple element s.)*

A geometric parameter is called a discrete series if in addition e is a distinguished nilpotent element.

The Springer and generalized Springer correspondences can be viewed as the analogue of Vogan's *lowest K-type* bijection for real groups between tempered $(\mathfrak{g}, K)$-modules with real infintesimal character and $\widehat{K}$.

Finally, we mention that for a geometric Hecke algebra, the unitarity of tempered modules also follows from the connection with the representations of the (dual) p-adic group.

3. Hecke algebras in the exceptional groups.

We will consider the example of $G = E_7$. We have the following bijection

$$\{(s,e,\psi)\}/G \leftrightarrow \widehat{\mathbb{H}(E_7)} \sqcup \mathbb{H}\widehat{(F_4, 1}, 2) \sqcup \kappa.$$

The first algebra is graded Hecke algebra with equal parameters of type E_7, the second one is of type F_4 with parameters 1 on the long roots, and 2 on the short roots, and κ is a cuspidal representation parametrized by the nilpotent $E_7(a_5)$. We use the Bala-Carter classification for nilpotent orbits, and the generalized Springer correspondence computed by Spaltenstein. The cuspidal local system to which $\mathbb{H}(F_4, 1, 2)$ is attached is on the Levi subalgebra $(3A_1)$" in E_7.

Definition 3.1. *A module U of $\mathbb{H}$ is called spherical if $Hom_W[triv : U] \neq 0$.*

We give an example in terms of Langlands classification, which illustrates the philosophy for classifying the unitary dual of $\mathbb{H}(F_4, 1, 2)$. Recall that we are only considering modules with real central character. Fix $\Pi_P \subset \Pi$, and let V be a tempered representation for $\mathbb{H}_M$. Not all tempered V need to be considered but only those which are in the *limit parameters.*

Proposition 3.2 (Barbasch-Moy)**.** *The Hermitian dual of $L(P, V, \nu)$ is $L(w_0P, w_mV, -w_0\nu)$, where w_m is an element of minimal length in $W_{w_0P} \cdot w_0 \cdot W_P$.*

We will assume from now on that (P, V, ν) is a Hermitian datum. Denote

$$\mathcal{U}_{\mathbb{H}}(P, V) = \{\nu : L(P, V, \nu) \text{ is unitary}\}.$$

This is a subset of $\mathfrak{a}$.

The tempered module V is parametrized by a Lie triple $\{e, h, f\}$. Let $\mathfrak{z}(e, h, f)$ denote the centralizer in $\mathfrak{g}$ of the Lie triple. Denote by $\mathbb{H}(P, V)$ the Hecke algebra attached to $\mathfrak{z}(e, h, f)$, with a parameter set c which is determined explicitly. (The parameters c are lengths of complementary series in the maximal parabolic cases.)

Principle.

$$\mathcal{U}_{\mathbb{H}}(P, V) \approx \mathcal{U}_{\mathbb{H}(P,V)}(0, triv) = \mathcal{SU}(\mathbb{H}(P, V)).$$

(The right hand side is the spherical unitary dual of $\mathbb{H}(P, V)$.)

Step 1. Nonunitarity. There is an intertwining operator

$$A(V, \nu) : I(P, V, \nu) \to I(P, V, -\nu)$$

which gives the Hermitian form on $L(P, V, \nu)$. For each $\mu \in \widehat{W}$, such that $Hom_{W_P}[\mu, V] \neq 0$, $A(V, \nu)$ induces an operator

$$A_\mu(V, \nu) : Hom_{W_P}[\mu : V] \to Hom_{W_P}[\mu : V].$$

These operators are normalized, so that $A_{\mu_0}(V, \nu) = 1$ on a W-type μ_0 (of multiplicity one) attached by the generalized Springer correspondence to the nilpotent element e. The condition for $L(P, V, \nu)$ to be unitary is that all operators $A_\mu(P, \nu)$ are positive semidefinite. In general, we actually use the reverse condition, so a condition for *nonunitarity.*

We need some explicit calculations in the maximal parabolic cases, but after that we can decide for which μ, $A_\mu(V, \nu)$ is the same as some spherical operator $A_\rho(\mu)(\nu)$ in $\mathbb{H}(P, V)$.

Step 2. Unitarity. We show that the remaining parameters ν are unitary by one of the following methods:

(1) irreducible deformations of unitarily induced modules from unitary parameters on subalgebras.

(2) IM-duals of unitary modules. The Hecke algebra has an involution which preserves unitarity (the *Iwahori-Matsumoto involution*). This is very hard to compute in general, so to apply it in our cases, we need to find composition series for endpoints of complementary series.

Example. In $\mathbb{H}(F_4, 1, 2)$, consider $\Pi_P = A_2$ (the long roots), and $V = St$. The nilpotent element is $e = A_2 + 3A_1$ in E_7. The centralizer is $\mathfrak{z} = G_2$. The principle says that $\mathcal{U}_{\mathbb{H}}(A_2, St) \approx \mathcal{SU}(G_2, 2, 1)$. In fact the answer is

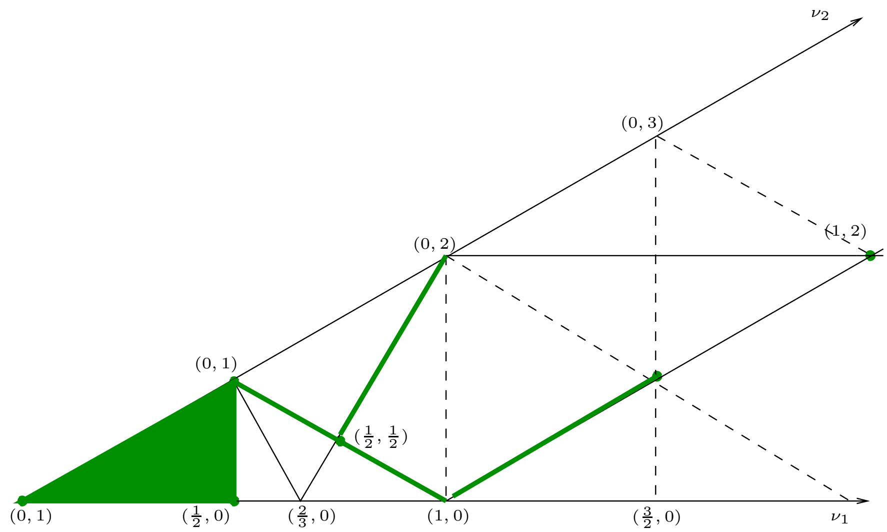

Figure 1. Unitary parameters and reducibility lines for $A_2 + 3A_1$

4. Illustration of the principle in type A

Let us consider the simplest example, that of the Hecke algebra of type A (necessarily with equal parameters, assumed to be all 1). The only discrete series is the Steinberg module. Moreover, since the induced $I(P, St, 0)$ is irreducible for any P, one can rewrite the Langlands classification as follows. The reason is that the centralizers of nilpotent elements in the adjoint group are all connected in type A.

Proposition 4.1. *Every irreducible $\mathbb{H}(A_n)$-module appears as the unique irreducible quotient $L(P, St, \nu)$ of a standard module $I(P, St, \nu) = \mathbb{H} \otimes_{\mathbb{H}_P} (St \otimes \mathbb{C}_\nu)$, where $\nu \in \mathfrak{a}^{\geq 0} = \{\nu \in \mathfrak{a} : \alpha(\nu) \geq 0,$ for all $\alpha \in \Pi_P\}$.*

If $L(P, St, \nu) \cong L(P', St, \nu')$ then $P = P'$ and $\nu = \nu'$.

The unitary dual in this case is of course a particular case of the work of Tadic. But it can be formulated via the principle stated before. (Since the only tempered module that appears in the standard modules notation is the Steinberg, we will drop St from these notations.)

In type A, the nilpotent orbits are uniquely determined in the Bala-Carter classification by subsets Π_P of Π, so denote by $\mathcal{O}_P$ the nilpotent orbit corresponding to P.

Conjecture 4.2. $\mathcal{U}_{\mathbb{H}}(P) = \mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}_P)))$.

Proof. We only sketch the proof of the inclusion $\mathcal{U}_{\mathbb{H}}(P) \subseteq \mathcal{SU}(\mathbb{H}(\mathfrak{z}(\mathcal{O}_P)))$. The general case reduces to the setting $\Pi = A_n$, $P = A_k^m$, $n = m(k+1)-1$. Then $\mathfrak{z}(\mathcal{O}_P)$ is of type A_{m-1}.

We use the simple roots $\Pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots\}$. The central character can be written as $s = (-\frac{k}{2} + \nu_1, \dots, \frac{k}{2} + \nu_1, \dots, -\frac{k}{2} + \nu_m, \dots, \frac{k}{2} + \nu_m)$. The Hermitian condition on s is the same as the Hermitian condition for the spherical parameter $\nu = (\nu_1, \dots, \nu_m)$ in A_{m-1}, i.e., $\nu_1 = -\nu_m$, $\nu_2 = -\nu_{m-1}$, etc. Assume this is satisfied.

The unique lowest W-type of $I(P, \nu)$ is $\mu_0 = (k+1, \dots, k+1) \otimes sgn$. (The tensoring with sgn comes from the normalization of the Springer correspondence that agrees with the Kazhdan-Lusztig classification.) The intertwining operators $A_\mu(\nu)$ are normalized so that $A_{\mu_0}(\nu) \equiv 1$. The relevant W-types for the spherical unitary dual of the centralizer (of type A_{m-1}) are $(m-l, l)$, $l \leq \frac{m}{2}$.

We identify the $W(A_n)$-types μ such that the operators $A_\mu(\nu) : Hom_{W_P}[\mu : St] \to Hom_{W_P}[\mu : St]$ are identical with the spherical operators on the relevant $W(A_{m-1})$-types. Explicitly, the matching is as follows

$$(\underbrace{k+2, \dots, k+2}_{l}, \underbrace{k+1, \dots, k+1}_{m-2l}, \underbrace{k, \dots, k}_{l}) \otimes sgn \leftrightarrow (m-l, l).$$

The calculation comes down to decomposing the operators $A_\mu(\nu)$ and an explicit maximal parabolic calculation (the case $m = 2$).

□

5. Maximal parabolic cases.

We consider the Hecke algebra with equal parameters. Let $\Pi_P \subset \Pi$ be maximal, $\{\alpha\} = \Pi - \Pi_P$, and $L(P, V, \nu)$ as in the Langlands classification. Let $\check{\omega}$ denote the fundamental coweight for α.

If $\mathfrak{p} = \mathfrak{m} + \mathfrak{n}$ is the corresponding maximal parabolic, V is attached to a map $\mathfrak{sl}(2) \hookrightarrow \mathfrak{m}$ (given by the Lie triple $\{e, h, f\}$ in the geometric parametrization of V). Then $\mathfrak{n}$ is an $\mathfrak{sl}(2)$-module, via the adjoint action of $\mathfrak{m}$. On the other hand, the coweight $\check{\omega}$ commutes with $\{e, h, f\}$, and $\mathfrak{n}$ decomposes as $\mathfrak{n} = \oplus_{i=1}^k \mathfrak{n}_i$, where $\mathfrak{n}_i$ is the i-eigenspace of $\check{\omega}$. Furthermore, decompose each $\mathfrak{n}_i$ into simple $\mathfrak{sl}(2)$-modules $\mathfrak{n}_i = \oplus_j (d_{ij})$, where by (d) we mean the simple $\mathfrak{sl}(2)$-module of dimension d. If (1) appears in the decomposition, denote by $i_0(P)$ the eigenvalue i for which it appears.

Example 1. We present first the case when the Langlands datum is generic.

Definition 5.1. *A module U of $\mathbb{H}$ is called generic if $Hom_{\mathbb{C}W}[sgn, U] \neq 0$.*

The result in the maximal parabolic generic cases is the following.

Theorem 5.2. *Let V be a generic discrete series of $\mathbb{H}_M$. Assume (P, V, ν) is Hermitian.*

(1) *If $\mathfrak{z}(e, h, f) = T_1$, $L(P, V, \nu)$ is unitary if and only if $\nu = 0$.*
(2) *If $\mathfrak{z}(e, h, f) = A_1$, then $L(P, V, \nu)$ is unitary if and only if*

$$0 \leq \nu \leq \frac{1}{i_0(P)}, \text{ if } (P, V) \neq (A_4 + A_2 + A_1, St) \text{ in } E_8;$$

$$0 \leq \nu \leq \frac{3}{10}, \text{ and } \nu = \frac{1}{2}, \text{ if } (P, V) = (A_4 + A_2 + A_1, St) \text{ in } E_8.$$

In conclusion, with the E_8 exception, $\mathcal{U}_{\mathbb{H}}(P, V) = \mathcal{SU}(\mathbb{H}(P, V))$.

Example 2. If we consider nongeneric Langlands data, we can get complementary series of "any length" (even for the Hecke algebra with equal parameters).

Let the nilpotent element e correspond to $(2, 4, \ldots, 2k)$ in $\mathfrak{sp}(2n)$. There are $\binom{k}{[\frac{k}{2}]}$ discrete series in the Hecke algebra (with equal parameters) attached to e. One of them, call it V_k, has a single W-type, μ_k, where $\mu_k \otimes sgn = \begin{cases} 0 \times (2m+1)^m, & k = 2m \\ (2m+1)^{m+1} \times 0, & k = 2m+1 \end{cases}$.

Proposition 5.3. *In $\mathbb{H}(C_{n+1})$, $L(C_n, V_k, \nu)$ is unitary if and only if $0 \leq \nu \leq [\frac{k}{2}] + \frac{1}{2}$.*

When we generalize these two examples to other parabolics (not maximal), we will find matchings with the spherical unitary dual of Hecke algebras (on centralizers) with equal, respectively unequal parameters. Therefore, even when we try to identify the unitary dual of the Iwahori-Hecke algebra, we are naturally led to consider (at least) the spherical unitary dual of Hecke algebras with unequal parameters.