David Vogan: Generalized induction

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1 Goal

The goal is to define a generalized induction functor

$$I : \left\{ \begin{array}{c} \text{virtual characters of} \\ \text{Levi subgroup} \end{array} \right\} \to \left\{ \begin{array}{c} \text{virtual characters of} \\ \text{real reductive group} \end{array} \right\}$$

and compute the action of I on irreducible modules using `atlas`.

If the Levi subgroup comes from a real (resp. θ stable parabolic) I should reduce to ordinary (resp. cohomological) parabolic induction.

2 Geometry

We fix the following notation:

$\mathcal{B} = \mathcal{B}(G)$: flag variety of Borel subgroups of a complex reductive group G.

$\mathcal{P}_S$: partial flag variety of parabolic subgroups of type $S \subset \Sigma$ (simple roots).

$\pi_S : \mathcal{B} \to \mathcal{P}_S$ − the natural projection.

Recall from Peter Trapa's lecture:

Given $K \subset G$ (fixed points of some involution), write $\texttt{kgb}(G, K)$ for the set of K orbits on $\mathcal{B}$. Then we have for each $s \in \Sigma$, an idempotent operation $\mathcal{O} \mapsto s(\mathcal{O})$ on $\texttt{kgb}(G, K)$. Here $s(\mathcal{O})$ is the unique open K-orbit in $\pi_{\{s\}}^{-1}\pi_{\{s\}}(\mathcal{O})$. We have $\mathcal{O} \subset \overline{s(\mathcal{O})}$; and either $\mathcal{O} = s(\mathcal{O})$ or $\dim(s(\mathcal{O})) = \dim(\mathcal{O}) + 1$.

Define $\sim_S$ to be the equivalence relation on $\texttt{kgb}(G, K)$ generated by $\mathcal{O} \sim s(\mathcal{O})$ for all $s \in S$. Each $\sim_S$ equivalence class has a unique maximal element, which has both largest length and greatest dimension. These distinguished maximal elements correspond to elements of `kgb` in which each root $s \in S$ is of type `c,r` or `C-` (cross action decreases length).

Problem 1 *Relate the structure of an individual $\sim_S$equivalence class C to* $\texttt{kgb}(L, M)$ *where $L = L_{S,C}$ is a Levi subgroup of G.*

Ideally we would like M to be the fixed points of some involution of L, but this is not always possible. However the problem will be solvable if we slightly enlarge the class of allowable subgroups M.

3 Easy example

Suppose G, K have equal rank, let C be an $\sim_S$ equivalence class containing a minimal element $\mathcal{O}$ of `kgb` (G, K). Then $\mathcal{O}$ is closed, and all simple roots are labeled `c` or `n` (they are all imaginary). Fix a Borel subgroup $B_{S,C}$ in $\mathcal{O}$, and let $Q_{S,C} = L_{S,C}U_{S,C}$ be the (θ-stable) parabolic subgroup of type S containing $B_{S,C}$. Then we get a map

$$\iota = \iota_C : \mathcal{B}(L_{S,C}) \to \mathcal{B}(G), \quad \iota(B_1) = B_1 U_{S,C}$$

Proposition 2 *The map ι gives a bijection between* `kgb` $\left(L_{S,C}, L_{S,C}^{\theta}\right)$ *and C.*

The map ι is compatible with cohomological induction I_C via $\mathfrak{q}_{S,C}$, and so we get a bijection

$$I_C : \{\text{irrs for } L_{S,C}\} \;\mapsto\; \{\text{irrs for } G \text{ supported on orbits in } C\}$$

Moreover, under I_C we have

$$\{\text{KL polynomials for } L_{S,C}\} \;\mapsto\; \{\text{KL polys for } G \text{ indexed by pairs in } C\}$$

4 Further example

We consider the group $U(2,1)$. In this case the `block` and `kgb` commands give the same output (all local systems are trivial). The Dynkin diagram and the idempotent action on the `block/kgb` (using `atlas` labels) looks like this:

$$\begin{array}{cccccccccccc}
 & & & & & & & 5 & & & & \\
 & & & & & & \nearrow s & & t \nwarrow & & & \\
s & — & t & & & 3 & & & & 4 & & \\
 & & & & t \nearrow & & \nwarrow t & & s \nearrow & & \nwarrow s & \\
 & & & 2 & & & & 0 & & & & 1
\end{array}$$

The closed orbits are $0, 1$ and 2. We fix $S = \{s\}$ corresponding to the first node in the Dynkin diagram. Then the maximal elements are $2, 4$ and 5, with equivalence classes $C_2 = \{2\}$, $C_4 = \{0, 1, 4\}$, $C_5 = \{3, 5\}$. The classes $C_2 = \{2\}$, C_4 are "easy" and correspond to the θ-stable Levi subgroups $L_2 = U(1) \times U(2)$ and $L_4 = U(1) \times U(1,1)$, respectively. The problem is to understand the equivalence class C_5.

We would like to have that

$$C_5 \text{ (and its corresponding block)} \longleftrightarrow \texttt{kgb}(L_5), \texttt{block}(L_5)$$

for some real form L_5 of $GL(2) \times GL(1)$. It seems at first that $GL(2, \mathbb{R}) \times U(1)$ might work since `kgb` $(GL(2, \mathbb{R}))$ looks like C_5. However `block` $(GL(2, \mathbb{R}))$ has three elements.

5 Generalizing the Atlas setup

What is really going in the previous example is that C_5 corresponds to the Bruhat order on the Weyl group $W = W\ (A_1)$. The structure involved is not that of "K"-orbits but rather "B"-orbits on the flag variety of $GL\,(2)$. To understand this in general, we need to extend the atlas setup to study the action of a more general class of subgroups on the flag variety.

Start with $G \supset B \supset T$, fix a parabolic subgroup $Q_I = L_I U_I$ corresponding to $I \subset \Sigma$, and a Cartan involution θ_I of L_I. Define $K_I = L_I^{\theta_I} U_I$. Thus $K_\Sigma =$ "K" while $K_\emptyset =$ "B". K_I acts on G/B with finitely many orbits; one can study the corresponding Harish-Chandra modules, localizations, KL polynomials etc.

Proposition 3 *One has*

$$\begin{aligned} \{K_I\text{-orbits on } G/B\} &\longleftrightarrow \{L_I^{\theta_I}\text{-orbits on } L_I/B\cap L_I\} \times \{Q_I\text{-orbits on } G/B\} \\ &\longleftrightarrow \texttt{kgb}\left(L_I, L_I^{\theta_I}\right) \times W/W_I \end{aligned}$$

Fix a K_I orbit $\mathcal{O}$ on G/B. Its $\sim_S$ equivalence class is $C = \{$orbits of K_I on $\pi_S^{-1}\left(\pi_S\left(\mathcal{O}\right)\right)\}$. This should be given as follows: Let B' be a Borel subgroup in $\mathcal{O}$ and let $Q_S = L_S U_S$ be the parabolic subgroup of type S containing B'.

Problem 4 *(Generalization of Problem 1)*

1. *Show that* $C \longleftrightarrow \{Q_S \cap K_I/U_S \cap K_I\text{-orbits on } L_S/B' \cap L_S\} \times W/W_S$
2. *Show that* $Q_S \cap K_I/U_S \cap K_I$ *is a "nice"* $K_J = L_J^{\theta_J} U_J \subset L_S$.
3. *Conclude that* $\sim_S$ *equivalence class of* $\mathcal{O}$ *in* $\{$*kgb or block for* $(G, K_I)\} \leftrightarrow \{$*kgb or block for* $\left(L_J, L_J^{\theta_J}\right)\} \times W/W_J$.

6 Induction functor

The natural bijection in the previous problem should be implemented on standard modules by an alternating sum of cohomological induction functors corresponding to the parabolic subalgebra $\mathfrak{q}_S$. To study this induction:

1. Start with an irreducible module X in a block of L_S at infinitesimal character $w\,(\rho_G)$ for $w \in W/W_L$.
2. Express X as a combination of standard modules using KL algorithm for L_S.
3. Apply induction to get a combination of standard modules for G at infinitesimal character ρ_G.
4. Express result in terms of irreducible modules for G, using KL for dual block of ${}^\vee G$.

It is possible (and desirable) to program `atlas` to implement this procedure.