Signatures of invariant Hermitian forms on irreducible highest weight modules and signed Kazhdan-Lusztig polynomials

wlyee@uwindsor.ca

G real red. $\supset K$ max'l cpt.

$\mathfrak{g}_0$ $\qquad$ $\mathfrak{k}_0$ $\qquad$ $\circlearrowleft \theta$ Cartan invol.

$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ etc.

Unitary dual problem:

Find $\hat{G}_u$ = equiv. classes of unitary irreps

(π, V) admiss. irrep. of G $\rightsquigarrow$ H.C.(π) Harish-Chandra module (admiss. $(\mathfrak{g}, K)$-mod.)

- $\pi \cong \pi'$ unitary $\Leftrightarrow$ H.C.$(\pi) \cong$ H.C.(π')
- unitary H.C. mod must be H.C.$\left(\begin{smallmatrix}\text{some}\\ \text{unitary rep}\end{smallmatrix}\right)$

$\hat{G}_u \longleftrightarrow$ H.C. mod's admitting pos. def'n inv. Herm. form

Zuckerman, '78: algebraic construction for admiss. $(\mathfrak{g}, K)$ mod's:

Cohomological induction

$\mathfrak{g} \supset \mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ θ-stable parabolic

$L = N_G(\mathfrak{q})$ Levi subgroup

$$\begin{tikzcd}
\mathcal{C}(\mathfrak{l}, L \cap K) \arrow[rr, "j^{th}\text{ cohom. ind.}", "\text{functor } \mathcal{L}_j"'] \arrow[dr, "\text{induction}"'] & & \mathcal{C}(\mathfrak{g}, K) \\
& \mathcal{C}(\mathfrak{g}, L \cap K) \arrow[ur, "j^{th}\text{ Bernstein functor}"'] &
\end{tikzcd}$$

V

$\mathfrak{u} \curvearrowright$ triv.

$\rightsquigarrow \mathcal{C}(\mathfrak{q}, L \cap K)$

$V \longmapsto \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V \longmapsto \Pi_j(\ldots)$

generalized Verma module

$\langle , \rangle$ inv. Herm. on V $\longmapsto$ $\langle , \rangle$ inv. Herm. on $\mathcal{L}_s V$

$s = \dim \mathfrak{u} \cap \mathfrak{k}$

Vogan '84: "antidom." V unitary $\longmapsto$ unitary $\mathcal{L}_s V$

Wallach '84: "antidom" V, $\langle , \rangle$ inv. Herm. signature $\longmapsto$ sig $\langle , \rangle$ on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} V$ $\longmapsto$ sig. $\langle , \rangle$ on $\mathcal{L}_s V$

Take $\mathfrak{g}$ minimal. We compute sig. of inv. Herm $\langle , \rangle$ on:

- irred. Verma mods $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$
  (remove "antidom" condition)
- irred. highest weight mods $L(\lambda)$

$$x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

Signature (p, q)

p = # pos, q = # neg eigenvalues for matrix rep. form

$M(\lambda)$ ∞-dim'l BUT:

$$\langle Hu, v \rangle + \langle u, \bar{H} v \rangle = 0$$

$$\Rightarrow M(\lambda)_\mu \perp M(\lambda)_\nu \text{ if } \nu \neq -\bar{\mu}$$

Decompose into fin. dim'l $\perp$ pieces:

$M(\lambda)_\mu$ — μ imaginary

$\underbrace{M(\lambda)_\mu \oplus M(\lambda)_{-\bar{\mu}}}_{\begin{pmatrix} 0 & A \\ \bar{A}^t & 0 \end{pmatrix}}$ — μ complex

#pos eigenvalues = #neg eigenvalues

Signature Character:

$$\sum_{\mu \text{ imaginary}} (p(\mu) - q(\mu)) e^\mu$$

inv. Herm form on $M(\lambda)$! up to $\mathbb{R}$ scalar

$\langle v_\lambda, v_\lambda \rangle_\lambda = 1$ Shapovalov form

det $\langle , \rangle_\lambda$ on $\lambda-\rho-\mu$ weight space:

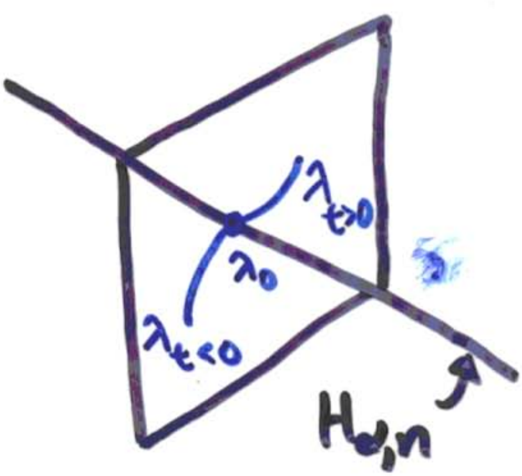
$$\prod_{\alpha \in \Delta^+} \prod_{n=1}^{\infty} \left((\lambda, \alpha^\vee) - n \right)^{P(\mu - n\alpha)}$$

Shapovalov det. formula $\rightsquigarrow$ reducibility

Reducibility hyperplane: $H_{\alpha,n} = \{ \lambda \mid (\lambda, \alpha^\vee) = n \}$

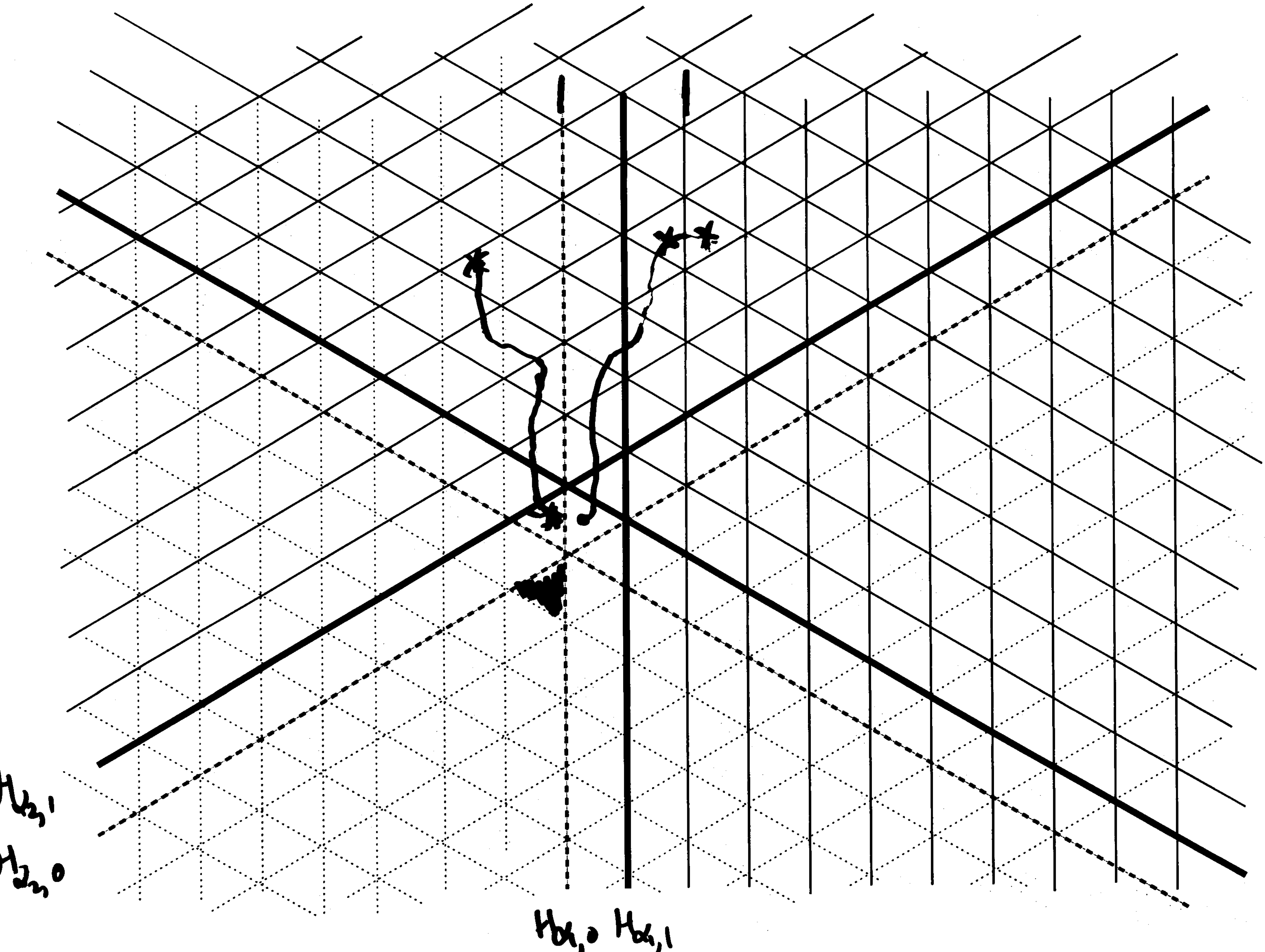
Connected component: sig can't change

↖ indexed naturally by W_a



$$\begin{aligned} \text{sig } t>0 &= \text{sig } t<0 \\ &\quad + \varepsilon \text{ sig Rad} \langle , \rangle_{\lambda_0} \\ &= \text{sig } t<0 \\ &\quad + \varepsilon \text{ sig } \langle , \rangle_{\lambda_0 - n\alpha} \end{aligned}$$

transl. $-n\alpha$ = refl. $H_{\alpha,0} \circ$ refl. $H_{\alpha,n}$



$H_{\alpha_2,1}$
$H_{\alpha_2,0}$
$H_{\alpha_1,0}$ $H_{\alpha_1,1}$

Formula phrased naturally in terms of affine Weyl group:

$W_a = W \ltimes \Lambda \rightsquigarrow$ homo $\bar{\cdot} : W_a \to W$

For $a \in W_a$ and $\tilde{a} \in W$ such that $aA_0 \subset \tilde{a}\mathfrak{C}_0$:

Theorem: Let $aA_0 = C_0 \xrightarrow{r_1} C_1 \xrightarrow{r_2} \cdots \xrightarrow{r_\ell} C_\ell = \tilde{a}A_0$ be a path from aA_0 to $\tilde{a}A_0$. Then for imaginary $\lambda \in A_0$:

$$ch_s M(\lambda) = \sum_{\substack{S=\{i_1<\cdots i_k\} \\ \subset\{1,\ldots,\ell\}}} \varepsilon(S) 2^{|S|} \frac{e^{\overline{r_{i_1} r_{i_2}} \cdots \overline{r_{i_k}} r_{i_k} r_{i_{k-1}} \cdots r_{i_1} \lambda - \rho}}{\prod_{\alpha \in \Delta^+(\mathfrak{p},\mathfrak{t})} (1 - e^{-\alpha}) \prod_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} (1 + e^{-\alpha})}$$

where $\varepsilon(S) = 1, -1, 0$ (see *"The signature of the Shapovalov form on irreducible Verma modules"* for computation of ε)

Irred. HWMs:

Rad $\langle , \rangle_\lambda = J(\lambda)$ largest proper submod.

$\langle , \rangle_\lambda$ on $M(\lambda) \rightsquigarrow$ non-deg. inv. Herm. $\langle , \rangle_\lambda$ on $L(\lambda) = M(\lambda) / J(\lambda)$

Signature?

Structure of $M(\lambda)$:

Composition Series:

$$V = V^0 \supset V^1 \supset V^2 \supset \cdots \supset V^N = \{0\}$$

V^i / V^{i+1} irred, called composition factors

Kazhdan-Lusztig Conjecture:

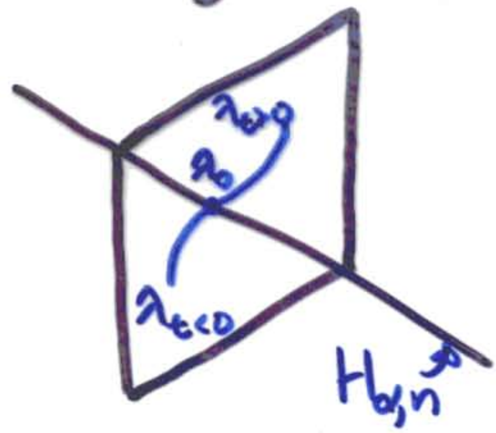
Assume λ antidom. integral, $x \in W$. Composition factors of $M(x\lambda)$ are of the form $L(y\lambda)$, $y < x$, with multiplicity $[M(x\lambda) : L(y\lambda)] = P_{w_0 x, w_0 y}(1)$

KL poly.

$$\operatorname{ch} M(x\lambda) = \sum_{y \in W} P_{w_0 x, w_0 y}(1) \operatorname{ch} L(y\lambda)$$

$$\operatorname{ch} L(x\lambda) = \sum_{y \in W} (-1)^{\ell(x) - \ell(y)} P_{y,x}(1) \operatorname{ch} M(y\lambda)$$

Idea: express sig $L(x\lambda)$ in terms of nearby Vermas



$M(\lambda_0)$ has composition factors

$$\boxed{\begin{array}{c} L(\lambda_0) \\ \hline L(\lambda_0 - n\alpha) \end{array}}$$

sig $t>0$: $\quad$ sig $L(\lambda_0) \pm$ sig $L(\lambda_0 - n\alpha)$

sig $t<0$: $\quad$ sig $L(\lambda_0) \mp$ sig $L(\lambda_0 - n\alpha)$

irred. Verma case

$$\frac{\text{sig } t>0 + \text{sig } t<0}{2} = \text{sig } L(\lambda_0)$$

More concretely: $\boxed{\text{Jantzen filtration}}$

- $\lambda_t = \lambda_0 + \delta t \qquad \delta \in \mathfrak{h}^*$ regular
- $\det \langle , \rangle_{\lambda_t} = 0$ only for $t=0$, t small

$$M = M(\lambda_0) = M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^N = \{0\}$$

$v \in M^j \iff \exists\, f_v : (-\varepsilon, \varepsilon) \to M$ s.t.

- $f_v(0) = v$
- $\langle v, v' \rangle_{\lambda_t}$ vanishes at least to order j at $t=0$

$\lim_{t\to 0^+} \frac{1}{t^j} \langle\,,\rangle_{\lambda_t}$ non-deg. inv. Herm. form $\langle\,,\rangle_j$ on $M_j := M^j / M^{j+1}$

sig (p_j, q_j)

Vogan '84:

$$\text{sig } t>0 = \left(\sum p_j, \sum q_j\right) \quad (*)$$

$$\text{sig } t<0 = \left(\sum_{j \text{ even}} p_j + \sum_{j \text{ odd}} q_j, \sum_{j \text{ even}} q_j + \sum_{j \text{ odd}} p_j\right)$$

$M(x\lambda)_j$ is semisimple: $\oplus\, L(y\lambda)$'s

$$[M(x\lambda)_j : L(y\lambda)] = \left[q^{\frac{\ell(x)-\ell(y)-j}{2}}\right] P_{w_0x, w_0y}$$

Sig $\langle\,,\rangle_j$ = sum of sigs of j^{th}-level $L(y\lambda)$'s

Introduce signed Kazhdan-Lusztig poly

Each $L(y\lambda)$ in $M(x\lambda)_j$ $\rightsquigarrow$ $+1, -1, 0$ to coeff. of $q^{\frac{\ell(x)-\ell(y)-j}{2}}$ in $P^{\lambda,\delta}_{w_0x, w_0y}$

signature info

$\rightsquigarrow +1$ to coeff ... in P_{w_0x, w_0y}

counting

$$\operatorname{sig}\langle\,,\rangle_j = \sum_{y\in W}\begin{pmatrix}\text{coeff. of } q^{\frac{\ell(x)-\ell(y)-j}{2}}\\ \text{in } P^{\lambda,\delta}_{w_0x,w_0y}\end{pmatrix}\operatorname{sig} L(y\lambda)$$

Sum over j, use ⊛

$$\operatorname{sig}\langle\,,\rangle_{x\lambda+\delta t} = \sum_{y\in W} P^{\lambda,\delta}_{w_0x,w_0y}(1)\,\operatorname{sig} L(y\lambda)$$

irred Verma

$(t>0)$

Invert:

$$\operatorname{sig} L(x\lambda) = \sum_{y_1<\cdots<y_j=x} (-1)^{j-1}\left(\prod_{i=2}^{j} P^{\lambda,\delta}_{w_0y_i,w_0y_{i-1}}(1)\right)\operatorname{sig}\langle\,,\rangle_{y_1\lambda+\delta t}$$

Want: Algorithm to compute $P^{\lambda,\delta}_{x,y}$

The **Kazhdan-Lusztig polynomials** are defined by:
$P_{x,x} = 1$, $P_{x,y} = 0$ when $x > y$ and:

a) $P_{w_0 x, w_0 y} = P_{w_0 xs, w_0 y}$ if $ys > y$ and $x, xs \geqslant y$, s simple

a') $P_{w_0 x, w_0 y} = P_{w_0 sx, w_0 y}$ if $sy > y$ and $x, sx \geqslant y$, s simple

b) If $y > ys$ and $x < xs$ then

$$P_{w_0 xs, w_0 y} + qP_{w_0 x, w_0 y} = \sum_{z \in W | zs > z} \mu(w_0 z, w_0 y) q^{\frac{\ell(z)-\ell(y)+1}{2}} P_{w_0 x, w_0 z} + P_{w_0 x, w_0 ys}$$

$$\begin{aligned} \text{where } \mu(w_0 z, w_0 y) &= \text{coeff of } q^{\frac{\ell(z)-\ell(y)-1}{2}} \text{ in } P_{w_0 z, w_0 y}(q) \\ &= [M(z\lambda)_1 : L(y\lambda)]. \end{aligned}$$

The **signed Kazhdan-Lusztig polynomials** are defined by:

$P_{x,x} = 1$, $P_{x,y} = 0$ when $x > y$ and:

a) $P^{\lambda,\delta}_{w_0x,w_0y} = \mathrm{sgn}(\delta, x\alpha)\varepsilon(H_{x\alpha,n}, xs)P^{\lambda,\delta}_{w_0xs,w_0y}$
if $ys > y$ and $x, xs \geqslant y$

a') $P^{\lambda,\delta}_{w_0x,w_0y} = \mathrm{sgn}(\delta, \alpha)\varepsilon(H_{\alpha,n}, sx)P^{\lambda,\delta}_{w_0sx,w_0y}$
if $sy > y$ and $x, sx \geqslant y$

b) If $x, y \in W$ are such that $x < xs$ and $y > ys$ and $x > y$ then:

$$\begin{aligned} &-(-1)^{\varepsilon((\lambda,\alpha^\vee)x\alpha)} P^{\lambda,\delta}_{w_0xs,w_0y}(q) + \mathrm{sgn}(\delta, x\alpha^\vee) q P^{\lambda,\delta}_{w_0x,w_0y}(q) \\ &= \sum_{z\in W_0 | z<zs} \mathrm{sgn}(\delta, z\alpha^\vee) a^{\lambda,\delta}_{w_0z,w_0y,1} q^{\frac{\ell(z)-\ell(y)+1}{2}} P^{\lambda,\delta}_{w_0x,w_0z}(q) \\ &\quad + \mathrm{sgn}(\delta, ys\alpha^\vee) P^{\lambda,\delta}_{w_0x,w_0ys}(q) \end{aligned}$$

where $a^{\lambda,\delta}_{w_0z,w_0y,1} = \text{coeff of } q^{\frac{\ell(z)-\ell(y)-1}{2}}$ in $P^{\lambda,\delta}_{w_0z,w_0y}(q)$.

Translation functors:

- ψ_α translation to α wall
- φ_α translation from α wall

Coherent continuation functor:

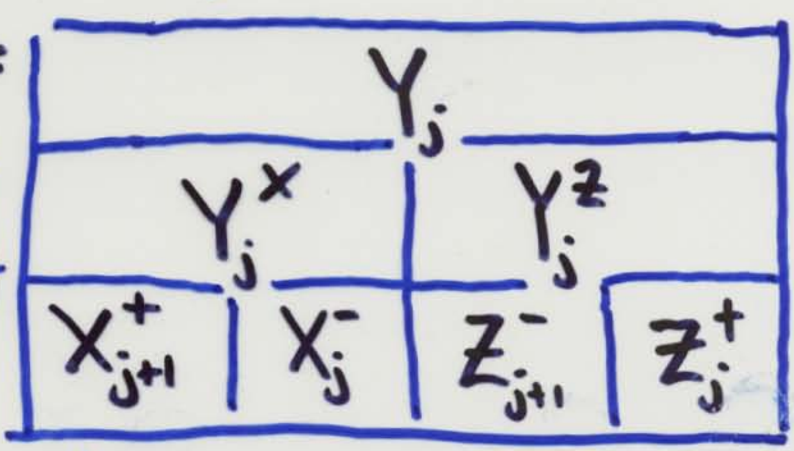
$$\theta_\alpha = \varphi_\alpha \circ \psi_\alpha : L(y\lambda) \mapsto \begin{cases} \text{non-zero} & y < ys_\alpha \\ 0 & y > ys_\alpha \end{cases}$$

For $x < xs_\alpha$:

$$\begin{array}{c} \theta_\alpha M(x\lambda) \\ \parallel \end{array}$$

$$0 \to \underset{X}{M(xs_\alpha\lambda)} \to \underset{Y}{\theta_\alpha M(xs_\alpha\lambda)} \to \underset{Z}{M(x\lambda)} \to 0$$

Four-step filtration:



$y < ys_\alpha$: $L(y\lambda)$ in + part

$y > ys_\alpha$: $L(y\lambda)$ in − part

X_{j+1}^+ paired with Z_j^+, so:

- $X_{j+1}^+ \cong Z_j^+$ +: case a)
- sig given by X_j^-, Z_{j+1}^-

Define $U_\alpha L(y\lambda)$ $(y < ys_\alpha)$ to be the cohomology of

$$0 \to L(y\lambda) \to \Theta_\alpha L(y\lambda) \to L(y\lambda) \to 0$$

Have SESes:

$$0 \to X_j^- \longrightarrow U_\alpha Z_j^+ \longrightarrow Z_{j+1}^- \longrightarrow 0$$

$$0 \to L(ys_\alpha\lambda) \to U_\alpha L(y\lambda) \to M(y\lambda)_1^- \to 0$$

X_j^-, Z_{j+1}^- : case b)

Gabber & Joseph's description of structure of $Y = \Theta_\alpha M(xs_\alpha\lambda)$, form on it

+

Jantzen's Determinant Formula

$\rightsquigarrow$ formula for signed Kazhdan-Lusztig polynomials

What now?

- of non-minimal, arbitrary
  - structure of GVM's?
    reducibility & comp. factors
  - extend signature formulas
- Kazhdan-Lusztig-Vogan algorithms
  - Verma — Standard; irred HWM — Langlands quotient
- other settings:
  - Virasoro algebra:
    classification of discrete series reps of Belavin-Polyakov-Zamolodchikov
    and Friedan-Qiu-Shenker
- signed Kazhdan-Lusztig polynomials:
  - $\supset$ usual KL-polys?
  - when a signed KL poly = KL poly, implications for unitarity?