Algorithms for Representation Theory

These are notes summarizing Jeff Adams' first talk at the winter meeting of the *Atlas of Lie Groups* project, March 2007. They briefly describe the combinatorics needed to make represetation theoretic calculations on a computer. These have been implemented in software by Fokko du Cloux.

1. Representation Theory

Representation theory is hard. We begin with a theorem:

Theorem 1.1. *Let G be a connected, complex, reductive algebraic group and let $G_{\mathbb{R}}$ be the real points of G. Let H be a Cartan subgroup of G and suppose that ρ exponentiates to H. Fix a regular, integral infinitesimal character $\lambda \in \mathfrak{h}^*$. Then there is a bijection:*

$$\begin{array}{c}\textit{Irreducible representations of } G_{\mathbb{R}} \\ \textit{w/ infinitesimal character } \lambda\end{array} \longleftrightarrow \{(H_{\mathbb{R}}, \chi)/G_{\mathbb{R}}\},$$

where $H_{\mathbb{R}}$ is a Cartan subgroup of $G_{\mathbb{R}}$, χ is a character of $H_{\mathbb{R}}$ whose differential is conjugate to λ, and the action of $G_{\mathbb{R}}$ on the set of pairs $(H_{\mathbb{R}}, \chi)$ is by conjugation.

Our goal is to find a combinatorial description of this set.

2. Combinatorics - The Tits Groups

Let R be a root system with Weyl group W and choice of simple roots Π. Form the group:

$$R^\vee/2R^\vee \cong (\mathbb{Z}_2)^n$$

and denote by m_α the element $\alpha^\vee + 2R^\vee$ for $\alpha \in \Pi$.

Definition 2.1. *The Universal Tits Group $\widetilde{W_u}$ is the group generated by $\{\sigma_\alpha \mid \alpha \in \Pi\}$ subject to the following relations:*

1. $\sigma^2 = m_\alpha$
2. $\sigma_{\alpha_i}\sigma_{\alpha_{i+1}}\sigma_{\alpha_i} = \sigma_{\alpha_{i+1}}\sigma_{\alpha_i}\sigma_{\alpha_{i+1}}$ *(Braid Relations)*
3. *Let $\gamma \in R$. Then $\sigma_\alpha \gamma^\vee {\sigma_\alpha}^{-1} = s_\alpha \cdot \gamma^\vee$, where the action on the right is the usual action of the Weyl group on $R^\vee$.*

The structure of $\widetilde{W_u}$ is given by the following theorem.

Theorem 2.2. *The following sequence is exact:*

$$1 \to R^\vee/2R^\vee \to \widetilde{W}_u \to W \to 1$$

In other words, $\widetilde{W}_u = W \times R^\vee/2R^\vee$ *as a set.*

We'd like to associate a similar group to our complex group G. To do this we choose a tuple $(G,\ H,\ B,\ \{X_\alpha\})$, where H is a Cartan subgroup of G, B is a Borel containing H, and $\{X_\alpha\}$ is a set of simple roots determined by B. Write $N = \mathrm{Norm}_G(H)$ for the normalizer of H in G and $W = N/H$ for the Weyl group. For each $\alpha \in \Pi$ choose a homomorphism

$$\phi_\alpha : \mathrm{SL}_2 \to G$$

such that

$$\phi_\alpha \begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} = \alpha^\vee(z)$$

and

$$\mathrm{d}\phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha.$$

Then we define:

$$\sigma_\alpha = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So for each $\alpha \in \Pi$ we have an associated $\sigma_\alpha \in N$. It turns out that the σ_α are uniquely determined for α simple, but they depend on our choices for $\{X_\alpha\}$. The σ_α can be thought of as representatives in N for the reflections $s_\alpha \in W$.

Definition 2.3. *The Tits Group for* G *is defined to be* $\widetilde{W} = <\sigma_\alpha \mid \alpha \in \Pi> \subseteq N$.

The following is an analog Theorem 2.2 for $\widetilde{W}$.

Theorem 2.4. *Let* $m_\alpha = {\sigma_\alpha}^2$ *and let* $H_0 = <m_\alpha> \subset H$. *Then*

(1) *There is an exact sequence*

$$1 \to H_0 \to \widetilde{W} \to W \to 1$$

(2) $\sigma_{\alpha_i}\sigma_{\alpha_{i+1}}\sigma_{\alpha_i} = \sigma_{\alpha_{i+1}}\sigma_{\alpha_i}\sigma_{\alpha_{i+1}}$ *(Braid Relations)*

(3) *There exists a unique set theoretic map* $s : W \to \widetilde{W}$ *taking a reduced expression in* W *to the corresponding expression in* $\widetilde{W}$.

The content of (3) is that the section s is well-defined.

3. The Algorithm

The starting datum for the algorithm will be a pair (G, γ), where G is our chosen complex group and γ is an inner class of real forms of G. Let θ_{fund} denote the involution corresponding to the fundamental form of the inner class γ. We then define

$$G^\Gamma = G \rtimes \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ acts on G via the involution θ_{fund}. We can now describe the main combinatorial objects in which we are interested.

Definition 3.1. $\widetilde{\mathcal{X}} := \{x \in Norm_{G^\Gamma \setminus G}(H) \mid x^2 \in Z(G)\}$ *where* $Z(G)$ *denotes the center of* G*.*

Definition 3.2. $\mathcal{X} := \widetilde{\mathcal{X}}/H$ *where* H *acts by conjugation.*

Remark 3.3. *If* G *is semi-simple, then the set* $\mathcal{X}$ *is finite.*

Example Let $G = \mathrm{SL}(2, \mathbb{C})$. Then there is a unique inner class of real forms and θ_{fund} is the identity map. The set $\mathcal{X}$ has five distinct elements up to H conjugacy. Here is a list of representatives:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Its clear from the definitions that the normalizer of H acts on the set $\mathcal{X}$. In fact, since H acts trivially (by definition) this action descends to W. The group G does not act on $\mathcal{X}$, however it still makes sense to consider the following set.

Definition 3.4. $\mathcal{X}[x] = \{y \in \mathcal{X} \mid y \sim x\} = G \cdot x \cap \mathcal{X}$.

For $x \in \mathcal{X}$ choose a lift $\eta \in \widetilde{\mathcal{X}}$ and define $\theta_\eta = \mathrm{int}(\eta)$. This is an involution of G and we define K_η to be the fixed points of θ_η, i.e. $K_\eta = G^{\theta_\eta}$. K_η is the complexification of a maximal compact subgroup for the (strong) real form of G corresponding to x. The following proposition explains our interest in the set $\mathcal{X}$.

Proposition 3.5. *The sets defined above have the following geometric interpretations (recall G/B is the flag variety):*

$$\mathcal{X}[x] \leftrightarrow K_\eta \backslash G/B$$

$$\mathcal{X} \leftrightarrow \coprod_{\eta \in \mathcal{I}} K_\eta \backslash G/B$$

where $\mathcal{I}$ is a set of representatives of elements in G that are lifts of elements in $\mathcal{X}$.

This proposition suggests that the set $\mathcal{X}$ is larger than we might like; it contains information about orbits on the flag variety for multiple (strong) real forms. However if we pick an $x \in \mathcal{X}$ then the set $\mathcal{X}[x]$ contains orbit information about a single real form of G. Thus our problem is to describe the set $\mathcal{X}[x]$. It turns out that the Tits group of G allows us to do this.

Fix $x \in \mathcal{X}$ and choose a lift $\eta \in \widetilde{\mathcal{X}}$. If we form the involution θ_η we can restrict it to H and get a Cartan involution independent of our lift η. Call this involution θ_x. Then θ_x gives a classification of the roots of H as real, imaginary, or complex as usual. Furthermore we obtain a grading on the imaginary roots (compact vs. noncompact) by using the involution θ_η. It turns out that this grading is independent of our choice of lift η.

Lemma 3.6. *Given $x \in \mathcal{X}$ it is possible to choose a lift η of x so that θ_η normalizes $\widetilde{W}$.*

We can now define the two crucial operations of $\widetilde{W}$ on $\mathcal{X}$.

(1) For $x \in \mathcal{X}$ and $\sigma_\alpha \in \widetilde{W}$ define the *cross action* of σ_α on x as follows:

$$\sigma_\alpha \cdot x := \sigma_\alpha \theta_\eta ({\sigma_\alpha}^{-1}) x,$$

where θ_η is given by the above lemma.

(2) Suppose α is noncompact imaginary for x (this is well-defined by the above discussion). Define the *Cayley transform* of x by α as follows:

$$x \mapsto \sigma_\alpha x$$

Remark 3.7. *It is not obvious that Cayley transforms preserve the set $\mathcal{X}[x]$. It is a pleasant fact that they do. This can be verified by an SL(2) calculation.*

We now come to the main theorem:

Theorem 3.8. *Let x be a fundamental element (i.e. over the fundamental Cartan). Then the set $\mathcal{X}[x]$ is equal to the set generated by x and successive cross actions and (when appropriate) Cayley transforms. In other words, the set $\mathcal{X}[x]$ can be computed inductively.*

From a computational standpoint, here is the algorithm. We start with x in the fundamental fiber and a list containing elements known to be in $\mathcal{X}[x]$ (initially just x). At each step we apply cross actions and (when appropriate) Cayley transforms to elements in the list and see if we get anything new. The content of Remark 3.7 is that anything new we find will belong to $\mathcal{X}[x]$. If we do find something new, we add it to the list and repeat the process. We continue until no new elements are found. By Theorem 3.8 this procedure finds every element of $\mathcal{X}[x]$. Thus we have computed the set of K_C orbits on the flag variety for the (strong) real form of G corresponding to x via the combinatorics of the Tits group. Amazing!